MODULE 5.
Z transform

In the last chapter we learned about Laplace transforms.

- Laplace transforms are mainly used to overcome the disadvantages of Fourier transforms.
- Fourier transforms cannot be used for unstable or unbounded systems.
- Fourier transform overcome this disadvantage by the use of Laplace transform.
- Laplace transforms overcomes the disadvantages of continous time Fourier transforms
- Similarly the disadvantages of

DTFT (Discrete time Fourier transform) can be solved by the use of z transform.

- Z transform is mainly used for discrete time signals

The Discrete time Fourier transform of a signal $x(n)$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

We get the transform in terms of the function $e^{j\omega}$.
that's why we use $X(e^{j\omega})$

The continuous time Fourier transform of a signal $x(n)$ is defined as

$$x(j\omega) \text{ or } X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

In discrete time FT, integration is replaced by summation and $e^{-j\omega t}$ is replaced by $e^{-j\omega n}$.

The existance condition of DTFT is given by

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

In case of CTFT

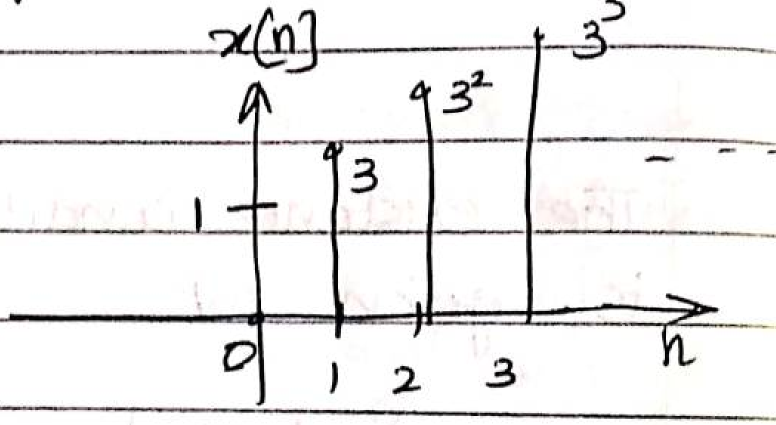
$$\int_{-\infty}^{\infty} |x(t)| \cdot dt < \infty$$

when this condition is failed we can't apply Discrete time Fourier transform.

- In case of unbounded or unstable system's we can't apply DTFT so, we go for z transform

- Let us consider a signal $x(n) = 3^n u(n)$. The schematic representation of the above signal is shown below
The above signal is a discrete time exponential signal. i.e, a growing exponential signal

$x(n) = 3^n u(n)$



So, from the figure, it is very clear that the above signal is an unbounded signal (Amp ↑ when time ↑)

- Here the signal $x(n) = 3^n u(n)$ is multiplied with r^{-n} where r^{-n} is a discrete time damping factor [in Laplace case, $e^{-\sigma t}$]

Now the new $x(n) = 3^n \cdot u(n) \cdot r^{-n}$

$= \left(3/r\right)^n \cdot u(n)$.

when $r = 1$ the above expression becomes $x(n) = 3^n \cdot u(n)$.

DTFT X

when $r = 2$, $x(n) = \left(3/2\right)^n \cdot u(n)$

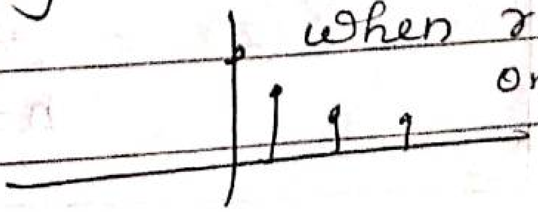
DTFT X

when $r = 3$ $x(n) = \left(3/3\right)^n u(n)$

DTFT X

when $r = 4$ $x(n) = \left(3/4\right)^n u(n)$

when $r > 3$ onwards we get a decaying discrete exponential signal.



when $r > 3$ onwards

Now from $r > 3$ onwards we can apply DTFT

Hence DTFT$[x(n) \cdot r^{-n}]$ —— (1)

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot r^{-n} \cdot e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (r e^{j\omega})^{-n}$$

The factor $r e^{j\omega} = z$.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n} \quad \text{—— (2)}$$

This is z transform of a signal $x[n]$.

$$\boxed{\begin{array}{l}\text{The z transform of a signal } x(n) \text{ is} \\ X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}\end{array}}$$

From ① & ② we get

$$X(z) = DTFT[x(n) \cdot r^{-n}]$$

Existance condition of z transform:

$$\sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$$

The factor $x(n) \cdot r^{-n}$ is absolutely summable. This is the existance condition of z transform.

The existance condition is not valid for all values of r.

- In the previous case, we get the transform when $r > 3$
- Therefore, ROC - Regeon of Convergence can be defined based on this

R.O.C → Region of convergence

The values of r for which the z transform of a Signal exists is known as Region of Convergence (ROC)

The term $z = re^{j\omega}$, which is of the form of polar representation where r denotes the magnitude of z, $|z|$ and ω denotes the phase angle.

- $r = |z|$

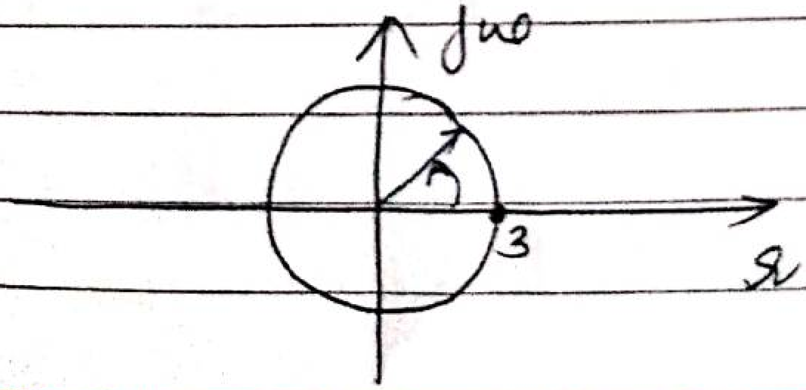
How to plot the ROC:

- Here ROC → the existance values (r values) plotted in z plane

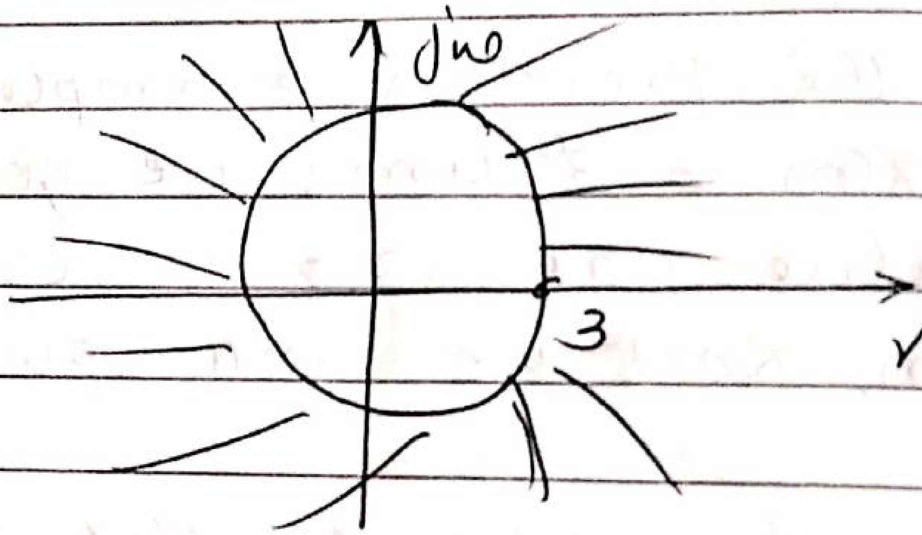
$z = re^{j\omega}$ → this is of the form of polar representation. So, we represent the ROC values in a circle. 'r' is the magnitude part of z. ω is the phase angle.

In the previous example $x(n) = 3^n u(n)$; we got R.O.C value as $r > 3$ or $|z| > 3$ then, how we will plot its ROC?

In order to plot its ROC, draw a circle of radius 3 in the z plane. X axis represents the magnitude (r) part and Y axis represents (jω) part



Here R.O.C. ~~|z|~~ $r>3$ or $|z|>3$
so, draw shaded portions
beyond $r>3$. so, the R.OC of
above signal will be of the
following form



This is the representation of
ROC of the signal $x[n] = 3^n \cdot u[n]$

Q. Find the R.O.C and z transform of $x[n] = \delta[n]$.

• Check the existance criteria

$$\sum_{n=-\infty}^{\infty} x(n) \cdot r^{-n} < \infty$$

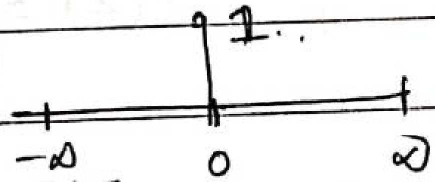
Here $x[n] = \delta[n]$

(We know that $x(n) \cdot \delta(n) = x(0) \cdot \delta[n]$

$$\sum_{n=-\infty}^{\infty} \delta[n] \cdot r^{-n}$$

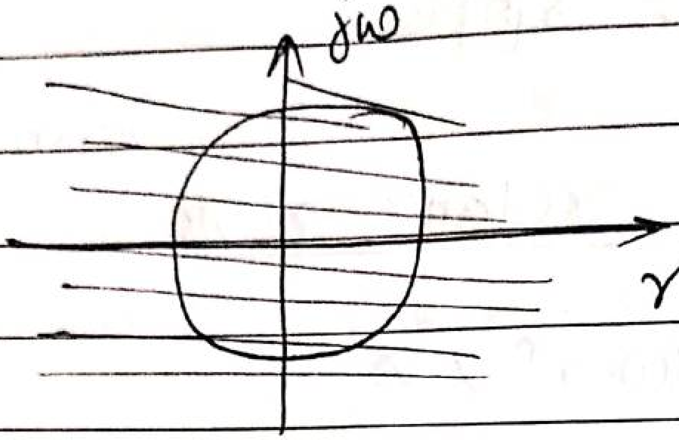
$$= \sum_{n=\infty}^{\infty} \delta[n] \cdot r^{-0}$$

$$= \sum_{n=\infty}^{\infty} \delta[n] = 1$$



Here the existance value we got is independent of r: so, we can say that the z transform of $x[n] = \delta[n]$ exists everywhere in the z plane.

The ROC of $x(n) = \delta(n)$ will be



Exists everywhere in the z plane

- find its transform value.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) \cdot z^{-0}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) = 1$$

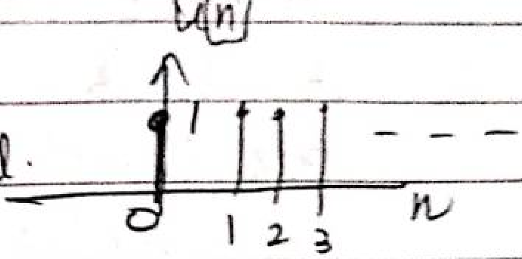
$\boxed{Z \cdot [\delta[n]] = 1}$

ROC : Exists everywhere in the z plane

Q. Find the Ztransform and ROC of
$x(n) = u[n]$.

$x[n] = u[n]$

unit step discrete time signal.



Existance condition

$$\sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$$

$$= \sum_{n=0}^{\infty} 1 \cdot r^{-n} < \infty$$

~~Put the values of n in r^{-n}~~

$=$

$$= r^{-0} + r^{-1} + r^{-2} + r^{-3} + \cdots \infty$$

$$= 1 + r^{-1} + r^{-2} + r^{-3} + \cdots \infty$$

$$= \frac{1}{1 - r^{-1}}$$

(This is of the form $1 + x + x^2 + \cdots \infty$

Ans $= \frac{1}{1-x}$ $|x| < 1$

Condition $r^{-1} < 1$

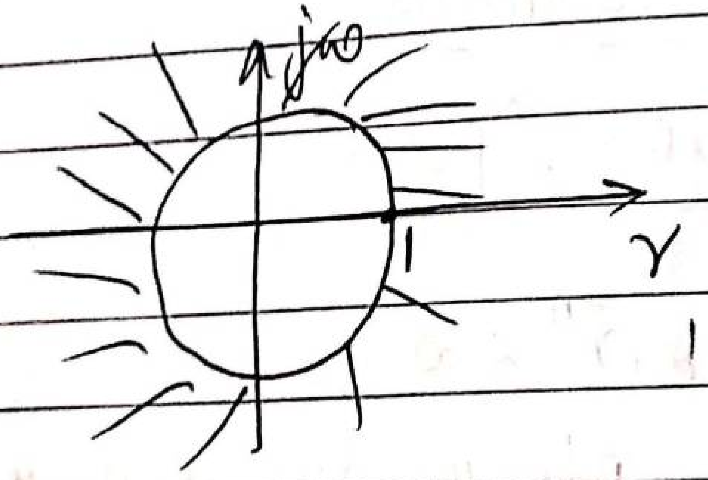
$$\frac{1}{r} < 1$$

$\underline{\underline{1 < r}}$ or $\underline{\underline{r > 1}}$ or $|z| > 1$

Scanned with CamScanner

Here the obtained ROC is $|z|>1$

To plot its ROC, draw a circle of radius 1. & draw shaded portions beyond $|z|>1$ to show the existance of the given signal



- Find transform value

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} u(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \infty$$

$$= \frac{1}{1-z^{-1}} \quad \text{or} \quad \frac{1}{1-\frac{1}{z}} = \frac{1}{\frac{z-1}{z}} = \frac{z}{z-1}$$

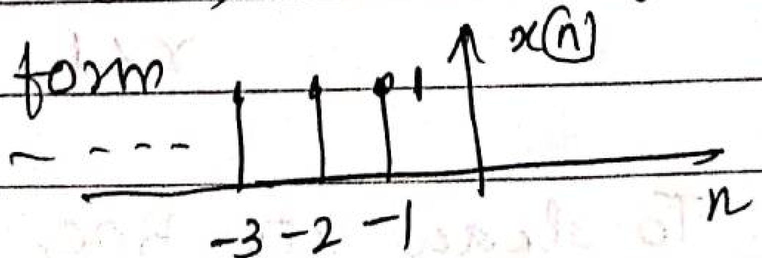
$$\boxed{Z[u(n)] = \frac{1}{1-z^{-1}} \quad \text{or} \quad \frac{z}{z-1}}$$

$$\boxed{\text{ROC} : |z| > 1}$$

3. Find the z transform and ROC of the signal

$x(n) = u(-n-1)$:

The signal $u(-n-1)$ will be of the following form



Existance criteria

$$\sum_{n=\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$$

$$= \sum_{n=-\infty}^{-1} 1 \cdot r^{-n}$$

(Change the limits from negative to positive, $\sum_{n=\infty}^{-1}$ then r^{-n} becomes r^{n}

$$= \sum_{n=1}^{\infty} r^{n}$$

$$\sum_{n=\infty}^{-1} = \sum_{n=1}^{\infty}$$

$$= r + r^2 + r^3 + r^4 + \cdots \infty$$

$$= r(1 + r + r^2 + r^3 + \cdots \infty)$$

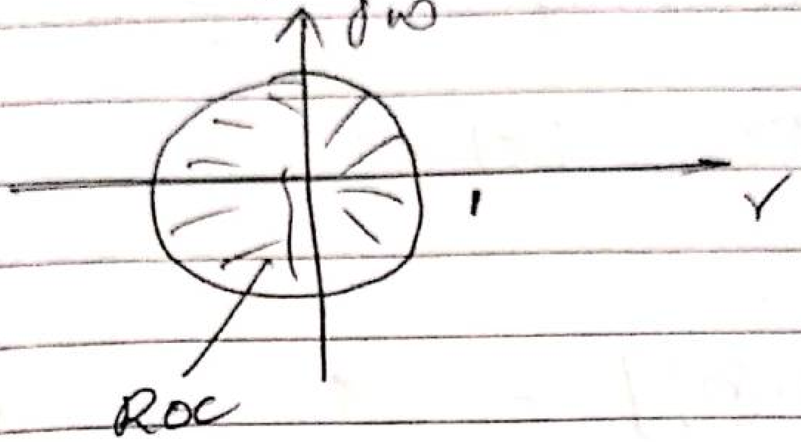
$$= r \cdot \left(\frac{1}{1-r} \right)$$

Here condition is

$r < 1$ or $|z| < 1$

To draw its ROC, draw a circle of radius of 1, and draw

Shaded regions inside the circle



- Find its value:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

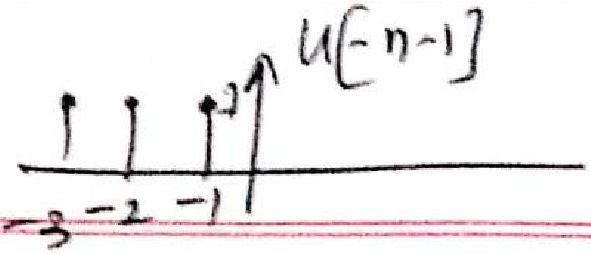
$$= \sum_{n=-\infty}^{-1} 1 \cdot z^{-n}$$

$$= \sum_{n=1}^{\infty} z^{n}$$

$$= z + z^2 + z^3 + - - \infty$$

$$= z \cdot [1 + z + z^2 + - \infty]$$

$$= \frac{z}{1-z} \quad \text{or} \quad -\left(\frac{z}{z-1}\right)$$

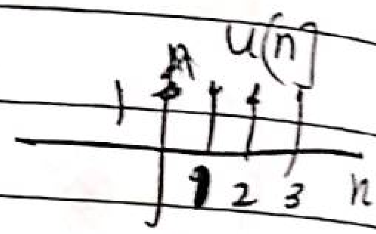


$$Z[u(-n-1)] = -\left(\frac{z}{z-1}\right)$$

ROC : $|z| < 1$

$$Z[u(n)] = \frac{z}{z-1}$$

ROC : $|z| > 1$



Q. Find the z transform and ROC of the signal $x(n) = a^n \cdot u(n)$

$x(n) = a^n \cdot u(n)$

Existance condition

$$\sum_{n=-\infty}^{\infty} |x(n) \cdot r^{-n}| < \infty$$

$$\sum_{n=0}^{\infty} |a^n \cdot 1 \cdot r^{-n}|$$

$$= \sum_{n=0}^{\infty} (a r^{-1})^n$$

$$= 1 + a r^{-1} + (a r^{-1})^2 + \cdots \infty$$

$$= \frac{1}{1 - a r^{-1}}$$

condition

$a r^{-1} < 1$

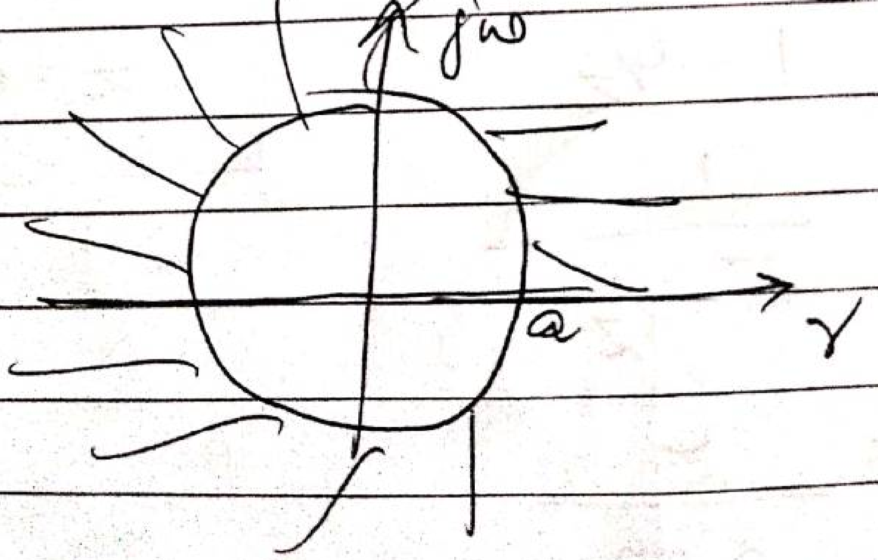
$a \cdot \frac{1}{r} < 1$

$\boxed{a < r \text{ or } \underline{r > a}}$

Plot ROC

Draw a circle of radius r

fits ROC is $r > a$



find its value

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + az^{-1} + (az^{-1})^2 + \cdots \infty$$

$$= \frac{1}{1-az^{-1}}$$

$az^{-1} < 1$

$a/z < 1$

$a < z$ or

$z > a$

$$= \frac{1}{1-a/z}$$

$$\frac{1}{\frac{z-a}{z}}$$

$$= \frac{z}{z-a}$$

$$Z\left[a^n u(n)\right] = \frac{1}{1-az^{-1}} \text{ or } \frac{z}{z-a}$$

ROC: $|z| > a$

Q. Determine the z transform of the signal $x(n) = -b^n \cdot u(-n-1)$. Find its ROC.

$x(n) = -b^n u(-n-1)$ is a left sided signal whose n value exists from $-\infty$ to -1

Existance condition

$$\sum_{n=-\infty}^{\infty} \left(x(n)\, r^{-n} \right) < \infty$$

$$= \sum_{n=-\infty}^{-1} -b^n \cdot r^{-n}$$

$$- \sum_{n=1}^{\infty} b^{-n} r^n$$

$$= -\sum_{n=1}^{\infty} (b^{-1}r)^n$$

$$= -[b^{-1}r + (b^{-1}r)^2 + + - \infty]$$

$$= -b^{-1}r \left[\frac{1}{1-b^{-1}r}\right]$$

Roc

$$\boxed{\begin{array}{c} b^{-1}r < 1 \ \text{OR} \\ r/b < 1 \\ \text{OR} \\ \underline{r < b} \end{array}}$$