PROPERTIES OF Z TRANSFORM

1. **Linearity**: If $x_1(n) \rightarrow X_1(z)$

$x_2(n) \rightarrow X_2(z)$

then $z[a x_1(n) + b x_2(n)]$

$= a X_1(z) + b X_2(z)$

Proof:

$$z(a x_1(n) + b x_2(n)]$$

$$= \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a x_1(n) . z^{-n} + \sum_{n=-\infty}^{\infty} b x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z)$$

2. TIME SHIFTING PROPERTY

If $x(n) \xrightarrow{z} X(z)$

then $x(n-m) \rightarrow z^{-m} X(z)$

Proof

$$Z[x(n-m)] = \sum_{n=-\infty}^{\infty} x(n-m) \cdot z^{-n}$$

Let $n-m = p$.

$$= \sum_{p=-\infty}^{\infty} x(p)\, z^{-(m+p)}$$

When $n=\infty$, $p=\infty-m$
$\underline{=\infty}$

When $n=-\infty$, $p=-\infty-m$
$= -\infty$

$$= \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-m} \cdot z^{-p}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p) \cdot z^{-p}$$

$$= \underline{\underline{z^{-m} \cdot X(z)}}.$$

③ Multiplication by an exponential Sequence:

If $x(n) \xrightarrow{Z} X(z)$

then $a^n x(n) \xrightarrow{Z} X(a^{-1} z)$

<u>proof</u>

$$Z[a^n \cdot x(n)] = \sum_{n=-\infty}^{\infty} a^n \cdot x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1} z)^{-n}$$

$$= X(a^{-1} z)$$

A) Time Reversal property

$$x(n) \xrightarrow{Z} X(z)$$

$$x(-n) \rightarrow X(z^{-1}) \text{ or } X(1/z)$$

<u>proof</u>

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) \cdot z^{-n}$$

Let $-n = p$

$$= \sum_{n=\infty}^{-\infty} x(p) \cdot z^{p}$$

$n = -\infty, \; p = \infty$

$n = \infty, \; p = -\infty$

$$= \sum_{n=-\infty}^{\infty} x(p) (z^{-1})^{-p}$$

$$= X(z^{-1})$$

5. Multiplication by n or differenciation in z domain

$x(n) \rightarrow X(z)$

$n\, x(n) \rightarrow -z \cdot \frac{d}{dz} X(z)$

Proof (~~RHS~~ first)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{n}$$

differentiate wrt z on both sides

$$\frac{d}{dz} X(z) = \frac{d}{dz} \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (-n)\, z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, (-n)\, z^{-n} \cdot z^{-1}$$

$$= \frac{-n}{z} \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$-z \cdot \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} (n\,x(n)) \cdot z^{-n}$$

$$= z(n\,x(n)]$$

$$\boxed{(n\,x(n)] \longrightarrow -z \cdot \frac{d}{dz} X(z)}$$

6 Convolution property

$$z(x(n) * h(n)] = X(z) \cdot H(z)$$

Proof

$$z(x(n) * h(n)] = \sum_{n=-\infty}^{\infty} (x(n) * h(n)]\, z^{-n} \quad -(1)$$

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k) \quad -(2)$$

Sub (2) in (1) then

$$z(x(n) * h(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} [x(k) \cdot h(n-k)]\right] z^{-n}$$

Interchange the order of integration

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot \sum_{n=-\infty}^{\infty} h(n-k) \cdot z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot ZT \cdot [h(n-k)]$$

$$= \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k} \cdot H(z).$$

$$= H(z) \cdot \sum_{k=-\infty}^{\infty} x(k) \cdot z^{-k}$$

$$= \underline{\underline{H(z) \cdot X(z)}}.$$

Time shifting property

$$z[x(n-m)] = z^{-m} X(z)$$

$$\underline{\underline{z[h(n-k)] = z^{-k} \cdot H(z)}}$$

PROBLEMS BASED ON PROPERTIES OF Z TRANSFORM

Find the z transform of the following signal using property

1. $x(n) = a^{n-1} \cdot u(n-1) \cdot \quad |z| > a$

$$z[a^n u(n)] = \frac{1}{1-az^{-1}} \text{ or } \frac{z}{z-a} \rightarrow X(z)$$

then $z[x(n-m)] = z^{-m} X(z)$

According to this property

$$z[a^{n-1} \cdot u(n-1)] = z^{-1} \cdot X(z)$$

$$= z^{-1} \, z[a^n \cdot u(n)]$$

$$= z^{-1} \times \frac{z}{z-a}$$

$$= \frac{1}{\cancel{z}} \times \frac{\cancel{z}}{z-a} = \frac{1}{z-a}$$

2. $x(n) = n \cdot a^n \cdot u(n)$

This is based on multiplication with n property

$$Z[n\,x(n)] = -z \cdot \frac{d}{dz}[X(z)]$$

$$Z[a^n u(n)] = \frac{z}{z-a}$$

then $Z[n\,x(n)] = -z \cdot \frac{d}{dz}[Z(n\,x(n))]$

$$= -z \cdot \frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z \cdot \left[\frac{(z-a)\,1 - z\cdot(1)}{(z-a)^2}\right]$$

$$= -z\left(\frac{\cancel{z} - a - \cancel{z}}{(z-a)2}\right)$$

$$= \frac{az}{(z-a)2}$$

classmate
Date
Page

$$z\left[\frac{a^n \cdot u(n)}{a}\right] = \frac{z}{(z-a)^2}$$

$$z\left(a^{n-1} \cdot u(n)\right] = \left(z/z-a\right)^2$$

3. Find the inverse z transform of $X(z) = \log(1-az^{-1})$ $|z|>a$

Here we use multiplication with n proper

$$z\left(n\,x(n)\right] = -z\frac{d}{dz}X(z)$$

$$\frac{d}{dz}\left(\log(1-az^{-1})\right) = \frac{1}{1-az^{-1}} \times (-1) \times (-a) z^{-2}$$

$$= \frac{az^{-2}}{1-az^{-1}}$$

$$-z \cdot \frac{d}{dz} X(z) = -z \cdot \left(\frac{az^{-2}}{1-az^{-1}}\right]$$

classmate

Date

Page

$$= -\cancel{z} \times a \times \frac{1}{z^2} \times \frac{1}{1-az^{-1}}$$

$$= \frac{-az^{-1}}{1-az^{-1}}$$

$$\therefore Z(n\,x(n)) = \frac{-az^{-1}}{1-az^{-1}}$$

$$= \frac{-a}{z} \times \frac{1}{1-a/z}$$

$$= \frac{-a}{\cancel{z}} \cdot \frac{\cancel{z}}{z-a}$$

$$Z(n\,x(n)) = \frac{-a}{z-a}$$

$$\therefore n\,x(n) = Z^{-1}\left[\frac{-a}{z-a}\right]$$

$$= -a\left[Z^{-1}\left(\frac{1}{z-a}\right)\right]$$

$$= -a \cdot a^{n-1} \cdot u(n-1)$$

$$= \underline{\underline{-a^{n} \cdot u(n-1)}}$$

Scanned with CamScanner

classmate

Date ______

Page ______

$$x(n) = \frac{-a^n}{n} \cdot [u(n-1)]$$

Q) find the inverse z transform of $X(z) = \log(1 - a^{-1}z)$ $|z| < a$.

$$Z(n\,x(n)) = -z \cdot \frac{d}{dz} X(z)$$

$$= -z \cdot \frac{d}{dz}\left[\log(1 - a^{-1}z)\right]$$

$$= -z \cdot \frac{1}{1 - a^{-1}z} \times a^{-1} \cdot 1]$$

$$= -z \times \frac{1}{1 - a^{-1}z} \times a^{-1}$$

$$= -z \times \frac{1}{a} \frac{1}{1 - z/a}$$

$$= -z/a \times \frac{a}{a - z}$$

$$= \frac{z}{z - a}$$

Scanned with CamScanner

Given ROC is $|z| < a$
$x(n) = -a^n \cdot u(-n-1)$

$$\{n\,x(n)\} = z^{-1}\left[\frac{z}{z-a}\right]$$

$$= \underline{\underline{-a^n \cdot u(-n-1)}}$$

$$\therefore x(n) = \underline{\underline{\frac{-a^n}{n} \cdot u(-n-1)}}$$

Q) find the inverse z transform

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) \qquad |z| < a$$

~~Q IZ of X(z) =~~

Q) Find the inverse z transform of $X(z) = \log\left(\frac{1}{1-a^{-1}z}\right)$ $\underline{\underline{|z| > a}}$

Scanned with CamScanner