$$= -z^{-1}\left\{ \frac{z}{z-a} \right\}$$

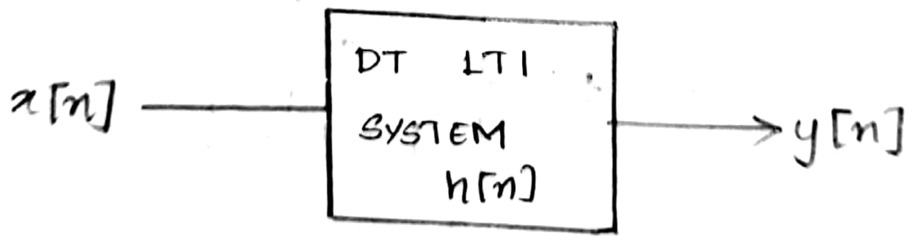
$$n\,x[n] = -a^{n}u[n]$$

$$x[n] = \frac{-a^{n}u[n]}{n}$$

23/04/2020

SOLUTION OF DIFFERENTIAL EQUATION USING Z- TRANSFORM

Consider a discrete time LTI system with input $x[n]$, its impulse response $h[n]$. The output of LTI system can be found out by using convolution.



$$y[n] = x[n] * h[n].$$

Applying z-transform on both sides, we get

$$Y(z) = X[z].H(z)$$

here $$\boxed{H(z) = \frac{Y(z)}{X(z)}}$$ H(z): system function (or) Transfer function.

(1) system function / Transfer function:

It is defined as ratio of z transform of the output to the z- transform of input under zero initial conditions

output of LTI system is given in terms of difference equation. Solution of difference equation is done with the help of z- transform.

(2). Impulse Response:

If H(z) is known, then the impulse function can be found out by taking inverse of system function or transfer function

$$h[n] = z^{-1}[H(z)] = z^{-1}\left[\frac{Y(z)}{X(z)}\right]$$

Impulse function is also found under zero Initial Conditions

step Response :

Step response is defined as the output of response from an LTI system when a unit step input is given.

From the system func: $H(z) = \frac{Y(z)}{X(z)}$

$$Y(z) = H(z).X(z)$$

Here $X(z)$ is the z-transform of unit step input.

Then $y[n] = z^{-1}\{Y(z)\}$

$x[n] = u[n]$

$$X(z) = \frac{z}{z-1}$$

$$= \frac{1}{1-z^{-1}}$$

Natural Response :

Natural response is response obtained by LTI system under initial conditions alone and zero input.

~~Natual~~ Natural Response = Apply Initial conditions | zero input.

Forced Response :

The response obtained from LTI system due to input alone and by considering initial conditions to zero.

Total response:
Response of LTI systems when both input and initial conditions are given.

Total Response = Apply Initial condition + Apply Input.

In order to solve the problem based on solution of difference equation by using z-transform the following conditions are very important.

when we neglect initial condition, in order to find the system function, impulse response, step response. The shifting property of z-transform is needed.

$$\left.\begin{aligned} Z\{x[n-m]\} &= z^{-m}X(z). \\ \text{Eg: } Z\{x(n-1)\} &= z^{-1}X(z) \end{aligned}\right\} \text{shifting property of bilateral z-transform.}$$

when we consider the initial conditions i.e to find the natural response, the total response (i/p + Initial conditions) the following equations are needed:

1. Time delay:

$$Z\{x(n-m)\} = z^{-m}X(z) + z^{-m}\sum_{k=1}^{\infty} x(-k)z^{k}.$$

$m = 1$

$$Z\{x(n-1)\} = z^{-1}X(z) + z^{-1}\sum_{k=1}^{1} x(-k)z^{k}.$$

$= z^{-1}X(z) + z^{-1} x(-1)z.$

$z\{x[n-1]\} = z^{-1}X(z) + x(-1).$

when $m = 2$.

$$z\{x(n-2)\} = z^{-2}X(z) + z^{-2}\sum_{k=1}^{2} x(-k)z^{k}.$$

$$= z^{-2}X(z) + z^{-2}\left[x(-1)z^{+1} + x(-2)z^{+2}\right]$$

$$= z^{-2}X(z) + x(-1)z^{-1} + x(-2)$$

$$\boxed{z\{x(n-2)\} = z^{-2}X(z) + x(-1)z^{-1} + x(-2)}$$

similarly;

$$z\{x(n-3)\} = z^{-3}X(z) + z^{-2}x(-1) + z^{-1}(x-2) + x(-3)$$

$$z\{x(n-4)\} = z^{-4}X(z) + z^{-3}x(-1) + z^{-2}(x-2) + z^{-3}(x(-3)) + x(-4)$$

Time Advance:

$$z\{x(n+m)\} = z^{m}X(z) - z^{m}\sum_{k=0}^{m-1} x(k)z^{-k}.$$

$m = 1$

$$z\{x(n+1)\} = zX(z) - z\sum_{k=0}^{0} x(k)z^{-k}.$$

$$= zX(z) - z.x(0)$$

$m = 2$

$$z\{x[n+2]\} = z^2X(z) - z^2\sum_{k=0}^{1} x(k)z^{-k}$$

$$= z^2X(z) - z^2\left[x(0)z^0 + x(1)z^{-1}\right]$$

$$= z^2X(z) - z^2x(0) - zx(1).$$

Similarly

$$z\{x[n+3]\} = z^3X(z) - z^3x(0) - z^2x(1) - zx(2).$$

Qn) Find the impulse response and step response for the following systems..

(1) $y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = x[n].$

Ans: Consider initial conditions to be 0.

Difference equation is given as

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}\{y[n-2]\} = x[n]$$

Applying z-transform on both sides.

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z)$$

Impulse response:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}.$$

$$H(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)}.$$

$$H(z) = \frac{A}{\left(1-\frac{z^{-1}}{4}\right)} + \frac{B}{\left(1-\frac{1}{2}z^{-1}\right)}.$$

$$A = \left.\frac{1}{1-\frac{1}{2}z^{-1}}\right|_{z^{-1}=4} \qquad B = \left.\frac{1}{1-\frac{1}{4}z^{-1}}\right|_{z^{-1}=2}.$$

$$= \frac{1}{1-2} \qquad = \frac{1}{1-\frac{1}{2}}$$

$$= \underline{\underline{-1}} \qquad = \underline{\underline{2}}$$

$$\therefore H(z) = \frac{-1}{\left(1-\frac{z^{-1}}{4}\right)} + 2\left(\frac{1}{1-\frac{z^{-1}}{2}}\right)$$

$$\therefore h[n] = -\left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[n]$$

Step Response :

Output $y[n]$ when $x[n] = u[n]$

$$x(z) = \frac{z}{z-1}$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{\left(1-\frac{z^{-1}}{4}\right)\left(1-\frac{z^{-1}}{2}\right)}$$

$$Y(z) = \frac{\left(\frac{z}{z-1}\right)}{\left(1-\frac{z^{-1}}{4}\right)\left(1-\frac{z^{-1}}{2}\right)}$$

$$= \frac{1}{(1-z^{-1})\left(1-\frac{z^{-1}}{4}\right)\left(1-\frac{z^{-1}}{2}\right)}$$

$$Y(z) = \frac{A}{(1-z^{-1})} + \frac{B}{\left(1-\frac{z^{-1}}{4}\right)} + \frac{C}{\left(1-\frac{z^{-1}}{2}\right)}$$

~~A = -1~~ ~~B = 2~~ (from previous impulse response)

$A_0 =$

$$A = \frac{1}{\left(1-\frac{z^{-1}}{4}\right)\left(1-\frac{z^{-1}}{2}\right)}\Bigg|_{z^{-1}=1}$$

$$= \frac{1}{\left(1-\frac{1}{4}\right)\left(1-\frac{1}{2}\right)}$$

$$= \frac{8}{3}.$$

$$B = \frac{1}{(1-z^{-1})(1-z^{-1}/2)}\Bigg|_{z^{-1}=4}$$

$$= \frac{1}{-3 \times -1}$$

$$= \frac{1}{3}.$$

$$C = \frac{1}{(1-z^{-1})\left(1-\frac{z^{-1}}{4}\right)}\Bigg|_{z^{-1}=2}.$$

$$= \frac{1}{(-1)\left(\frac{1}{2}\right)} = \underline{\underline{-2}}.$$

$$\therefore Y(z) = \frac{8}{3}\left[\frac{1}{1-z^{-1}}\right] + \frac{1}{3}\left[\frac{1}{1-\frac{z^{-1}}{4}}\right] + - 2\left[\frac{1}{1-\frac{z^{-1}}{2}}\right]$$

$$\therefore y[n] = \frac{8}{3}(1^n)u[n] + \frac{1}{3}\left(\frac{1}{4}\right)^n u[n] - 2\left[\frac{1}{2}\right]^n u[n].$$

Homework:

(2) $y[n] = x[n] + 2x[n-1] - 4[x[n-2]] + x[n-3]$

Ans: Consider Initial conditions to be zero.

Taking inverse transform on both sides of the equation:

$$Y(z) = X(z) + 2z^{-1}X(z) - 4z^{-2}X(z) + z^{-3}(X(z)$$

$$\frac{Y(z)}{X(z)} = (1 + 2z^{-1} - 4z^{-2} + z^{-3})$$

Impulse Response:

$$H(z) = (1 + 2z^{-1} - 4z^{-2} + z^{-3})$$

comparing with $\{x(0), x(1)z^{-1}, x(2)z^{-2}, \ldots x(\infty)z^{-\infty}\}$

we get $h[n] = \{\underset{\uparrow}{1}, 2, -4, 1\}$

$\Downarrow$

Impulse signal

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step_response.

$y[n] = ?$ when $x[n] = u[n]$.

$$Y(z) = X(z)\left[1 + 2z^{-1} - 4z^{-2} + z^{-3}\right]$$

$$X(z) = \frac{1}{1-z^{-1}}$$

$$= \frac{1}{(1-z^{-1})}\left[1 + 2z^{-1} - 4z^{-2} + z^{-3}\right]$$

$$= \frac{1}{1-z^{-1}} + 2\left[\frac{z^{-1}}{1-z^{-1}}\right] - 4\left[\frac{z^{-2}}{1-z^{-1}}\right] + \frac{z^{-3}}{(1-z^{-1})}$$

$$u[n] \stackrel{z}{\longleftrightarrow} \frac{1}{1-z^{-1}} \qquad u[n-2] \longleftrightarrow z^{-2}\left[\frac{1}{1-z^{-1}}\right]$$

$$u[n-1] \stackrel{z}{\longleftrightarrow} z^{-1}\left[\frac{1}{1-z^{-1}}\right] \qquad u[n-3] \longleftrightarrow z^{-3}\left[\frac{1}{1-z^{-1}}\right]$$

$$\therefore\ y[n] = u[n] + 2u[n-1] - 4\left[u[n-2]\right] + u[n-3].$$

(Qn) Find transfer function and impulse response of the system described by the difference eq.

$$y[n] - \frac{1}{3}y[n-1] = 4x[n-1].$$

Ans: ~~Fo~~ for Initial conditions $=0$.

Taking z-transform on both sides.

$$Y(z) - \frac{1}{3} z^{-1} Y(z) = 4 z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{4z^{-1}}{1 - \frac{z^{-1}}{3}}$$

$$H(z) = \frac{4z^{-1}}{1 - \frac{z^{-1}}{3}}$$

$$h[n] = 4\left(\frac{1}{3}\right)^{n-1} u[n-1]$$

Qn) Determine impulse response & step response of causal system given below.

A $\quad y[n] - y[n-1] - 2y[n-2] = x[n-1] + 2x[n-2]$

Ans: Initial conditions = 0.

Taking z- transform.

$$Y(z) - z^{-1}Y(z) - 2z^{-2}Y(z) = z^{-1}(X(z)) + 2z^{-2}X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{z^{-1} + 2z^{-2}}{1 - z^{-1} - 2z^{-2}}$$

$-2 \; -1 + 1$

$$\cancel{H(z) = \frac{A}{(z^{-1}+1)} + \frac{B}{(z^{-1}-2)}}$$

$$\frac{\frac{z+2}{z^2}}{1 - z^{-1} - 2z^{-2}}$$

$$\frac{z^{-1} + 2z^{-2}}{(z+1)(z-2)} = \cancel{\frac{A(z-2) + B(z+1)}{(z+1)(z-2)}}$$

$$\frac{z+2}{z^2 - z - 2}.$$

$$H(\cancel{z}) = \frac{A}{1+\cancel{z^{-1}}} + \frac{B}{1-\cancel{2z^{-1}}}$$

$$H(z) = \frac{z+2}{z^2-z-2}$$

$$= \frac{z+2}{(z+1)(z-2)}$$

$$\frac{z+2}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}.$$

$$\frac{z+2}{\cancel{(z+1)(z-2)}} = \frac{A(z-2)+B(z+1)}{\cancel{(z+1)(z-2)}}$$

$$z+2 = A(z-2) + B(z+1)$$

$z=-1$

$1 = A(-1)$

$A = -1$

$z = 2.$

$3B = 5$

$B = 5/3.$

$$\therefore H(z) = -\left[\frac{1}{z+1}\right] + \frac{5}{3}\left[\frac{1}{z-2}\right].$$

$$H(z) = \frac{z^{-1} + 2z^{-2}}{1 - z^{-1} - 2z^{-2}}.$$

$$H(z) = \frac{z^{-1} + 2z^{-2}}{(1+z^{-1})(1-2z^{-1})}$$

$$H(z) = \frac{A}{1+z^{-1}} + \frac{B}{1-2z^{-1}}.$$

$$A = \left.\frac{z^{-1} + 2z^{-2}}{1 - 2z^{-1}}\right|_{z^{-1} = -1}$$

$$= \frac{-1+2}{1+2}.$$

$$= \frac{1}{3}.$$

$$B = \left.\frac{z^{-1} + 2z^{-2}}{1 + \cancel{2}z^{-1}}\right|_{z^{-1} = \frac{1}{2}}$$

$1 - 2z^{-1} = 0$

$2z^{-1} = 1$

$z^{-1} = 1/2.$

$$= \cancel{\frac{1/2 + 2/4}{1 - 2/4}} \quad \frac{1/2 + 1/2}{1 + 1/2}.$$

$$= \frac{1}{3} \times 2$$

$$= 2/3.$$

$$H(z) = \frac{1}{3}\left[\frac{1}{1+z^{-1}}\right] + \frac{2}{3}\left[\frac{1}{1-2z^{-1}}\right].$$

$\therefore\ h[n] = \frac{1}{3}\left[(-1)^n u[n]\right] + \frac{2}{3}\left[2^n u[n]\right].$

step Response:

$y[n] = ?$ when $x[n] = u[n]$

$X(z) = \frac{1}{1-z^{-1}}.$

$$Y(z) = X(z) \cdot H(z)$$

$$= \frac{1}{1-z^{-1}}\left[\frac{z^{-1} + 2z^{-2}}{(1+z^{-1})(1-2z^{-1})}\right].$$

$$Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1+z^{-1}} + \frac{C}{1-2z^{-1}}$$

$$A = \left.\frac{z^{-1} + 2z^{-2}}{(1+z^{-1})(1-2z^{-1})}\right|_{z^{-1}=1}$$

$$= \frac{1+2}{2(1-2)}$$

$$= \frac{-3}{2}.$$

$$B = \left.\frac{z^{-1} + 2z^{-2}}{(1-z^{-1})(1-2z^{-1})}\right|_{z^{-1}=-1}$$

$$= \frac{-1+2}{(1+1)(1+2)} = \frac{1}{2 \times 3} = \frac{1}{6}$$

$$C = \left. \frac{z^{-1} + 2z^{-2}}{(1-z^{-1})(1+z^{-1})} \right|_{z^{-1} = 1/2}.$$

$$= \frac{1/2 + 2(1/4)}{(1 - 1/2)(1 + 1/2)}.$$

$$= \frac{1}{(1/2)(3/2)}$$

$$= \frac{4}{3}.$$

$$\therefore Y(z) = -\frac{3}{2}\left[\frac{1}{1-z^{-1}}\right] + \frac{1}{6}\left[\frac{1}{1+z^{-1}}\right] + \frac{4}{3}\left[\frac{1}{1-2z^{-1}}\right]$$

$$y[n] = -\frac{3}{2}\left[1^n u[n]\right] + \frac{1}{6}\left[(-1)^n u[n]\right] + \frac{4}{3}\left[2^n u[n]\right]$$

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