MODULE 6

DISCRETE TIME FOURIER TRANSFORM

- Discrete time fourier transform of a signal $x(n)$ is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

- Discrete time FT can be applied only for discrete time signals

- Continous time FT of a signal $x(t)$ is

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

- The existance condition of CTFT is $\int_{-\infty}^{\infty} x(t) \cdot dt < \infty$

- The existance condition of DTFT is $\sum_{n=\infty}^{\infty} x(n) < \infty$

- Discrete time fourier transform & Z transform are interrelated

Relation b/w z transform and Discrete time Fourier transform

The z transform of a signal $x(n)$ is

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

by Putting $z = e^{j\omega}$, the above expression changed in to DTFT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot (e^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}.$$

ie DTFT of $x(n)$ = z transform of $x(n)\big|_{z=e^{j\omega}}$

$$\boxed{X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}}}$$

This is the Relation between z transform and Discrete time Fourier transform.

Simple method to Calculate DTFT

1. If a signal $x(n)$ is given, and the existance criteria of signal is satisfied. Then we find the z transform of that signal. After that we put $z = e^{j\omega}$ to get its DTFT

Q. Find the DTFT of $x(n) = 2^n \cdot u(n)$.

Existance criteria is not satisfied
This is an unbounded signal $(a>1)$
So, DTFT of that signal doesn't exist

Q. Find the DTFT of $x(n) = (1/2)^n \cdot u(n)$

Existance condn is satisfied

$$X(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}$$

$z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}}$$

put $\omega = \omega + 2\pi$

$$X(e^{j(\omega+2\pi)}) = \frac{1}{1-\frac{1}{2}e^{-j(\omega+2\pi)}}$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}\cdot e^{-j2\pi}}$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}} = X(e^{j\omega})$$

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$X(e^{j\omega})$ is periodic with period of $\underline{\underline{2\pi}}$.

Here in DTFT, signal $x(n)$ is discrete in time domain, and its transform is periodic in frequency domain with period 2π. It's frequency spectrum is continuous also

The inverse FT of discrete time signal.

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \cdot e^{j\omega n} \cdot d\omega$$

The inverse FT of CT signal is

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

Here the frequency domain is not periodic, i.e. why we are taking $-\infty$ to ∞.

Q. Find the DTFT of the following signals

1. $x(n) = \delta(n)$. Z transform $X(z) = 1$

$z = e^{j\omega}$ $X(e^{j\omega}) = 1$

2) $x(n) = u(n)$. $X(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$

$= \frac{1}{1-z^{-1}}$

$z = e^{j\omega}$

$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}}$

3) $x(n) = u(n-k)$ $X(z) = \sum_{n=k}^{\infty} 1 \cdot z^{-n}$

$= z^{-k} + z^{-(k+1)} + z^{-(k+2)} + \cdots$

$= z^{-k}\left[1 + z^{-1} + z^{-2} + \cdots\right]$

$= z^{-k}\left[\frac{1}{1-z^{-1}}\right]$

$z = e^{j\omega}$

$X(e^{j\omega}) = e^{-j\omega k} \cdot \frac{1}{1-e^{-j\omega}}$

4. $x(n) = a^n \cdot u(n)$.

$$X(z) = \frac{1}{1-az^{-1}}$$

$z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{1}{1-ae^{-j\omega}}$$

5 $x(n) = -a^n \; u(-n-1)$

$$X(z) = \frac{1}{1-az^{-1}}$$

$$= \frac{1}{1-ae^{-j\omega}}$$

6. $x(n) = (-a)^n \cdot u(n)$

$$X(z) = \frac{1}{1+az^{-1}}$$

$z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{1}{1+ae^{-j\omega}}$$

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7. $x(n) = \delta(n-k)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{k} \delta(n-k) \cdot z^{-n}$$

$n=k$.

$\delta(n-k)=0$

$z^{-n} = z^{-k}$

$$= z^{-k}$$

$z = e^{j\omega}$

$$X(e^{j\omega}) = e^{-j\omega k}$$

Shifted value of impulse multiplied by a signal equal to that signal with 'n' replaced with the existance value of $\delta(n)$

Here $\delta(n-k) = 1$ at $n=k$

$\delta(n-k)$

k

So: $\delta(n-k) \cdot z^{-n}$

z^{-k}

8. $x(n) = \delta(n+2) - \delta(n-2)$

$X(z) = e^{j\omega}$

$$= z^{2}(1) - z^{-2}(1)$$

Put $z = e^{j\omega}$

$$X(e^{j\omega}) = e^{2j\omega} - e^{-2j\omega}$$

$$= 2j \sin\omega$$

use the shifting property of z

$z(x(n+m)) = z^{m} \cdot X(z)$

$z(x(n-m)) = z^{-m} \cdot X(z)$

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9. Find the DTFT of the following signals.

1. $x(n) = \{1, -1, 2, 2\}$

$X(z) = 1 - z^{-1} + 2z^{-2} + 2z^{-3}$

$z = e^{j\omega}$

$X(e^{j\omega}) = 1 - e^{-j\omega} + 2e^{-2j\omega} + 2e^{-3j\omega}$

2. $x(n) = 2^n . u(n)$

$2^n u(n)$ is not absolutely summable

$a > 1 \Rightarrow$ exp rising signal. so we can't find its DTFT.

$$x(n) = (.5)^n u(n) + 2^n u(-n-1)$$

$$X(z) = \sum_{n=0}^{\infty} (.5)^n z^{-n} + \sum_{n=-\infty}^{-1} 2^n z^{-n}$$

$$X(z) = \sum_{n=0}^{\infty} (.5 z^{-1})^n + \sum_{n=1}^{\infty} 2^{-n} z^{n}$$

$$= \sum_{n=0}^{\infty} (.5 z^{-1})^n + \sum_{n=1}^{\infty} (2^{-1} z)^n$$

$$= \frac{1}{1-.5z^{-1}} + 2^{-1} z \left[\frac{1}{1-2^{-1}z}\right]$$

Put $z = e^{j\omega}$

$$X(e^{j\omega}) = \frac{1}{1-.5e^{-j\omega}} + \frac{.5e^{j\omega}}{1-.5e^{j\omega}}$$

$$= \frac{1 - \cancel{.5e^{j\omega}} + \cancel{.5e^{-j\omega}} - .25\underbrace{e^{-j\omega} e^{j\omega}}_{1}}{1-.5e^{j\omega} - .5e^{-j\omega} + .25}$$

$$= \frac{1-.25}{1-.5(e^{j\omega}+e^{-j\omega})+.25} = \frac{.75}{1-2\times.5\cos\omega+.25}$$

$$= \frac{.75}{1.25-\cos\omega}$$

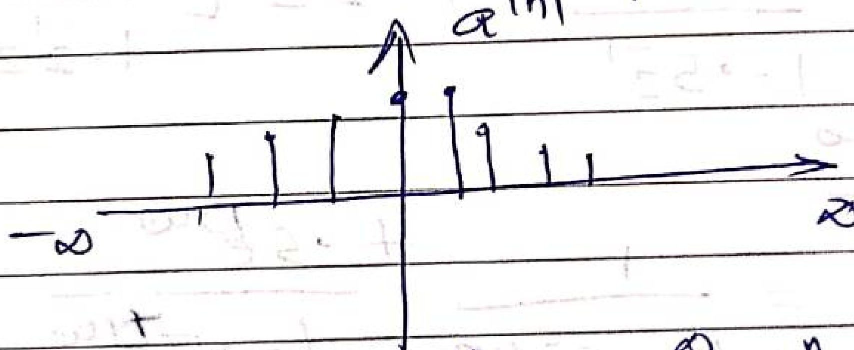
Q. Find the Fourier transform of the signal

$$x(n) = a^{|n|}$$

$a^{|n|}$ is defined as

$$x(n) = \begin{cases} a^{n} & \text{for } n>0 \\ a^{-n} & \text{for } n<0. \end{cases}$$

It's schematic representation is



$$X(z) = \sum_{n=-\infty}^{-1} a^{-n} \cdot z^{-n} + \sum_{n=0}^{\infty} a^{n} \cdot z^{-n}$$

$$= \sum_{n=1}^{\infty} a^{n} z^{n} + \sum_{n=0}^{\infty} (a z^{-1})^{n}$$

$$= az\left[\frac{1}{1-az}\right] + \frac{1}{1-az^{-1}}$$

Put $z = e^{j\omega}$

$$X(e^{j\omega}) = a e^{j\omega} \cdot \frac{1}{1-a e^{j\omega}} + \frac{1}{1-a e^{-j\omega}}$$

on cross multiplying, we get

$$\frac{ae^{j\omega}(1-ae^{-j\omega}) + 1-ae^{j\omega})}{(1-ae^{j\omega})\cdot(1-ae^{-j\omega})}$$

$$= \frac{ae^{j\omega} - a^2 + 1 - ae^{j\omega}}{1-ae^{-j\omega} - ae^{j\omega} + a^2}$$

$$= \frac{1-a^2}{1-2a\cos\omega + a^2}$$