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Discrete Time Fourier Transform properties

1. Linearity :

In z - transform

$$a[x_1[n]] + b[x_2[n]] \longrightarrow aX_1(z) + bX_2(z)$$

substitute. $z = e^{jw}$.

$$a\,x_1[n] + b\,x_2[n] \xrightarrow{DTFT} aX_1(e^{jw}) + bX_2(e^{jw})$$

Proof :

$$x[n] \xrightarrow{z} X_1(z)$$

$$x_1[n] \longrightarrow \sum_{n=-\infty}^{\infty} x_1[n] z^{-n}$$

$$= X_1(z)$$

$$z = e^{jw}$$

$$x_1[n] = X_1(e^{jw}) \quad —①$$

Similarly. $x_2[n] \longrightarrow X_2(e^{jw})$. — ②

$a \times ① + b \times ② \Rightarrow$

$$a\,x_1[n] + b\,x_2[n] \rightarrow \cancel{aX_1(z) + b}\ a\,X_1(e^{jw}) + bX_2(e^{jw})$$

2. Time shifting :

$$x[n] \rightarrow X(e^{jw})$$

$$\text{DTFT}\{x[n-m]\} \rightarrow e^{-jwm} \cdot X(e^{jw})$$

proof.

$$x[n] \xrightarrow{z} X(z)$$

$$x[n-m] \rightarrow \sum_{n=-\infty}^{\infty} x[n-m] z^{-n}$$

let $n-m=k$.

$n = k+m$.

when $n=\infty$ $k=\infty$

$n=-\infty$ $k=-\infty$

$$\rightarrow \sum_{k=-\infty}^{\infty} x[k] z^{-(k+m)}$$

$$z^{-m} \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

$$z^{-m} X(z)$$

sub: $z = e^{jw}$.

$$(e^{jw})^{-m} \cdot X(e^{jw}).$$

$$\therefore x[n-m] \xrightarrow{DTFT} e^{-jwm} \cdot X(e^{jw})$$

3. Frequency shifting:

$$DTFT[x[n] \cdot e^{-j\omega_0 n}] \rightarrow X[e^{j(\omega-\omega_0)}]$$

$$x[n] \xrightarrow{DTFT} X(e^{j\omega})$$

$$x[n]e^{+j\omega_0 n} \xrightarrow{DTFT} \sum_{n=-\infty}^{\infty} x[n]e^{+j\omega_0 n} \cdot e^{-j\omega n}$$

$$\cancel{\sum_{n=-\infty}^{\infty} x[n]} e^{-j[\omega_0 \cancel{+} \omega]n}$$

$$\rightarrow \sum_{n=-\infty}^{\infty} x[n] e^{j[\omega_0-\omega]n} = \sum_{n=-\infty}^{\infty} x[n] e^{-j[\omega-\omega_0]n}$$

$$\Rightarrow X(e^{j(\omega-\omega_0)n})$$

$$\therefore x[n]e^{j\omega_0 n} \xrightarrow{DTFT} \cancel{\sum_{n=-\infty}^{\infty}} X(e^{j(\omega-\omega_0)n}$$

4. ~~Tir~~ Time Reversal:

$$x[n] \xrightarrow{DTFT} X(e^{j\omega})$$

$$x[-n] \xrightarrow{DTFT} X(e^{-j\omega})$$

Proof:

$$x[-n] = \sum_{n=-\infty}^{\infty} x[-n]e^{-j\omega n}$$

$$= \sum_{-n=-\infty}^{\infty} x[n]e^{j\omega n}$$

$$= \sum_{-n=-\infty}^{\infty} x[n] e^{-(j\omega(-n))}$$

$$= \underline{\underline{X(e^{-j\omega})}}$$

$$x[-n] \xrightarrow{DTFT} X(e^{-j\omega})$$

5. Differentiation in frequency :

$$x[n] \xrightarrow{DTFT} X(e^{j\omega})$$

$$n x[n] \xrightarrow{DTFT} j \frac{d}{d\omega} X[e^{j\omega}]$$

$$X(e^{j\omega}) = DTFT\{x[n]\}.$$

$$= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

Differentiating both sides.

$$\frac{d}{d\omega}[X(e^{j\omega})] = \sum_{n=-\infty}^{\infty} \frac{d}{d\omega}[x[n] e^{-j\omega n}]$$

$$\frac{d}{d\omega}[X(e^{j\omega})] = \sum_{n=-\infty}^{\infty} x[n] \left[-jn e^{-j\omega n}\right]$$

$$\frac{-1}{j} \frac{d}{d\omega}[X(e^{j\omega})] = \sum_{n=-\infty}^{\infty} n x[n] e^{-j\omega n}.$$

$$\therefore \quad n[x[n]] \xrightarrow{DTFT} j \frac{d}{d\omega}[X(e^{j\omega})]$$

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6. Convolution in time domain:

$$F[x_1[n] * x_2[n]] = X_1(e^{j\omega}) \cdot X_2(e^{j\omega})$$

Proof:

$$x_1[n] \xrightarrow{DTFT} X_1(e^{j\omega})$$

$$x_2[n] \xrightarrow{DTFT} X_2(e^{j\omega})$$

$$x_1[n] * x_2[n] \xrightarrow{DTFT} \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k) e^{-j\omega n}$$

$$\therefore \cancel{x_1 * x} \; x_1[n] * x_2[n] \rightarrow \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x(k) . h(n-k)$$

$$x_1[n] * x_2[n] \xrightarrow{DTFT} \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) h(n-k) e^{-j\omega n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \underbrace{\sum_{n=-\infty}^{\infty} h(n-k) e^{-j\omega n}}_{\text{Time shifting property.}}$$

$$= \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k} H(e^{j\omega})$$

$$= He^{j\omega} \sum_{k=-\infty}^{\infty} x(k) e^{-j\omega k}$$

$$x_1[n] * x_2[n] \xrightarrow{DTFT} \underline{\underline{X(e^{j\omega}) . He^{j\omega})}}$$

7. **Parseval's Theorem:**

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega .$$

Proof:

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} x[n]\, x^*[n].$$

$$= \sum_{n=-\infty}^{\infty} x[n] \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \right]^* d\omega .$$

$$= \cancel{\frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{n=-\infty}}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega}) \underbrace{\sum_{n=-\infty}^{\infty} x[n]\, X(e^{-j\omega})}_{X(e^{j\omega})} \, d\omega .$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, X(e^{j\omega}) \, d\omega .$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega$$

hence proved ...