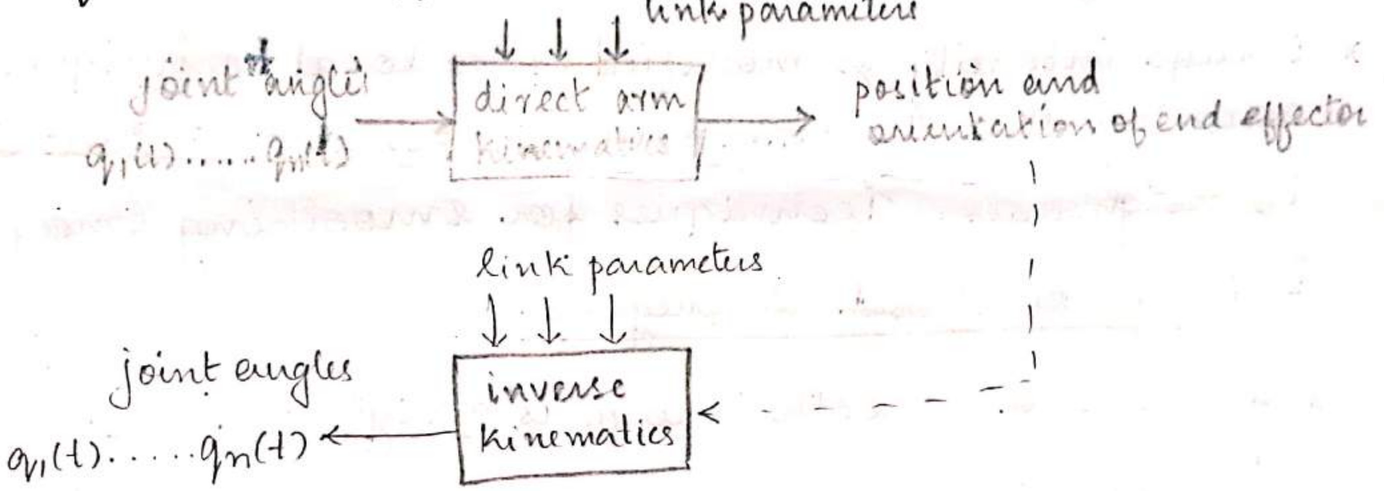
KINEMATICS -

* deals with the analytical study of the geometry of motion of a robot arm w.r.t a fixed reference coordinate frame as a fn of time without regard to the forces / moments that cause the motion.
* deals with analytical description of the spatial displacement of the robot as a fn of time ie: relations b/w joint variable space and the posn & orientation of the end-effector of a robot arm.



① Given joint angle vector, $q(t) = (q_1(t), q_2(t) \ldots q_n(t))^T$ and geometric link parameters, where $n \to$ no: of degrees of freedom, we can find position and orientation of the end-effector of the manipulator w.r.t a reference coordinate s/m. [direct / forward]

② Given posn & orientation of end-effectors and geometric link parameters, w.r.t a reference coordinate s/m we can find manipulator hand position and orientation. [inverse / arm soln]

∴ independent variables in a robot arm are joint variables and a task is stated in terms of reference coordinate frame.

* Links may rotate / translate w.r.t a reference coordinate frame, total spatial displacement of the end-effector is due to the angular rotations and linear translations of the links.

* Denavit and Hartenberg [1955] - systematic and generalized approach of utilizing matrix algebra
* uses a 4×4 homogenous transformation matrix and it is used to describe the spatial relationship b/w 2 adjacent rigid links and reduces direct kinematics problem to find an eq of
✓ 4×4 homogenous transformation matrix that relates spatial displacement of hand coordinate frame to the reference coordinate frame
* useful in deriving the dynamic eqns of motion of a robot arm.

* Inverse kinematics problem can be solved by several techniques.
* Matrix algebraic, iterative / geometric approaches.
* Geometric approach is based on the link coordinate systems and the manipulator configurations
* General approach using 4×4 homogenous matrices

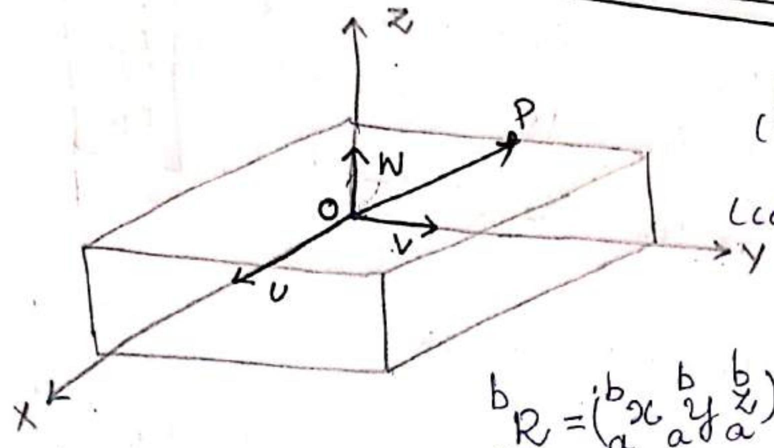
Direct Kinematics Problem

∵ the links of a robot arm may rotate / translate w.r.t a ref: coordinate frame, a body attached coordinate frame 'll be established along the joint axis for each link.

* Direct Kinematics problem is reduced to finding a transformation matrix that relates body attached coordinate frame to the reference coordinate frame.
* A 3×3 rotation matrix - describe rotational operations of the body-attached frame w.r.t reference frame.
* Homogeneous coordinates are then used to represent position vectors in a 3D space, rotation matrices will be expanded to 4×4 homogeneous transformation matrices to include the translational operations of the body attached coordinate frames.
* Advantage of D-H repn of linkages is its algorithmic universality in deriving kinematic eqn of a robot arm.

Rotation Matrix (relative orientation of 2 such frames)

* transformation matrix which operates on a position vector in a 3D euclidean space and maps its coordinates expressed in a rotated coordinate s/m OUVW (body attached frame) {b} to a ref: coordinate s/m OXYZ



position of frame's (vector) origin

Orientation of its axes (co-ordinalisim) relative to ref: frame

6DOF ⇒ in 3D space
3 Rot + 3 Translation

$${}^{b}_{a}R = ({}^{b}_{a}x \; {}^{b}_{a}y \; {}^{b}_{a}z)$$

$\{a\}$

OXYZ – fixed in 3D space and its reference frame.
OUVW – rotating w.r.t reference frame, OXYZ.
i_x, j_y, k_z – coordinate axes of OXYZ.
i_u, j_v, k_w – " OUVW.

* A point P in space is represented by its coordinates w.r.t both coordinate s/m.

* P – rest and fixed w.r.t OUVW coordinate frame.

$P_{uvw} = (P_u \, P_v \, P_w)^T$ and $P_{xyz} = (P_x \, P_y \, P_z)^T$ T – transpose.

A <u>3 x 3 transformation matrix, R</u> which transforms coordinates of P_{uvw} to P_{xyz}.

$$P_{xyz} = R \, P_{uvw}$$

$P_{uvw} = P_u i_u + P_v j_v + P_w k_w$ <u>Defining component of a vector</u>

$P_x = i_x \cdot P = i_x \cdot i_u p_u + i_x \cdot p_v j_v + i_x \cdot p_w k_w$

$P_y = j_y \cdot P = j_y \cdot i_u p_u + j_y \cdot p_v j_v + j_y \cdot p_w k_w$

$P_z = k_z \cdot P = k_z \cdot i_u p_u + k_z j_v p_v + k_z \cdot k_w p_w$.

In matrix form,

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} i_x i_u & i_x j_v & i_x k_w \\ j_y i_u & j_y j_v & j_y k_w \\ k_z i_u & k_z j_v & k_z k_w \end{bmatrix} \begin{bmatrix} P_u \\ P_v \\ P_w \end{bmatrix}$$

$$R = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_v & i_x \cdot k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_x \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix} \rightarrow (1)$$

Inverse:

$$p_{uvw} = Q P_{xyz}.$$

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} i_u i_x & i_u j_y & i_u k_z \\ j_v \cdot i_x & j_v j_y & j_v k_z \\ k_w i_x & k_w j_y & k_w k_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \rightarrow (2)$$

$$Q = \begin{bmatrix} i_u \cdot i_x & i_u \cdot j_y & i_u \cdot k_z \\ j_v \cdot i_x & j_v \cdot j_y & j_v \cdot k_z \\ k_w \cdot i_x & k_w \cdot j_y & k_w \cdot k_z \end{bmatrix} \rightarrow (3)$$

∵ Dot products are commutative

$$Q = R^{-1} = R^T$$

$$QR = R^T R = R^{-1} R = I_3$$

I_3 – identity matrix

$$\left.\begin{array}{l} P_{xyz} = R p_{uvw} \\ P_{uvw} = Q P_{xyz} \end{array}\right\} \text{orthogonal transformation}$$

and since the vectors in the dot products are all unit vectors, it is also called orthonormal transformation

* to find the rotation matrices that represent rotations of the OUVW coordinate s/m abt OXYZ.

* If OUVW coordinate s/m is rotated an α angle abt OX axis to arrive at a new location in the space, then the point p_{uvw} will have diff: coordinates w.r.t OXYZ s/m.

coordinate s/m

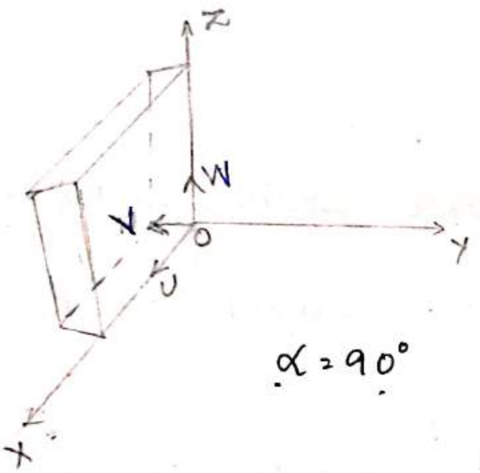
Necessary transformation matrix $R_{x,\alpha}$ is called the rotation matrix abt OX axis with α angle.

$$P_{xyz} = R_{x,\alpha} P_{uvw} \text{ with } i_x = i_u$$

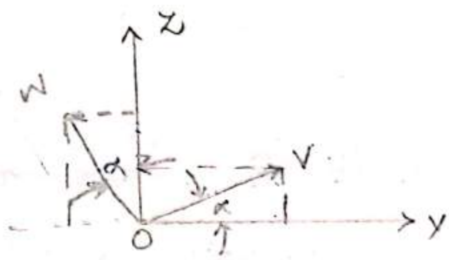
$$R_{x,\alpha} = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_v & i_x \cdot k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$\cos(90+\alpha)$, $\cos(90-\alpha)$

nox, noU



$\alpha = 90°$



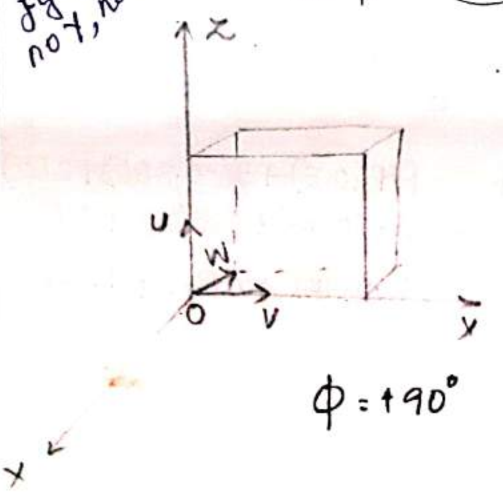
Rotation matrix abt OY axis with ϕ angle

$$R_{y,\phi} = \begin{bmatrix} i_x \cdot i_u & i_x j_v & i_x k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix}$$

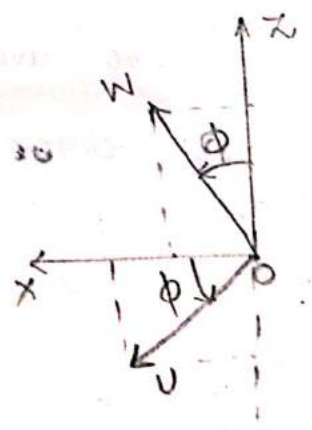
$\cos(90-\phi)$, $\cos(90+\phi)$

$j_y = j_v$, noY, noV

noY, noV

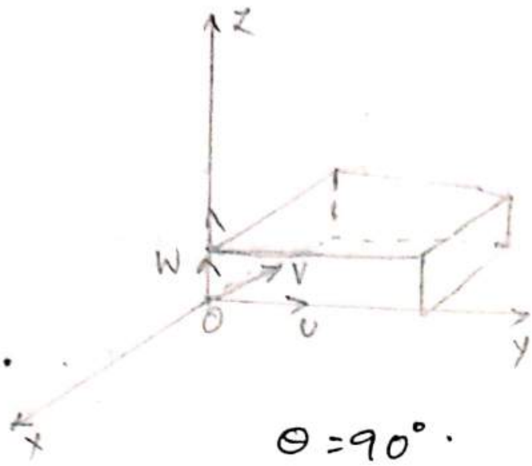


$\phi = +90°$

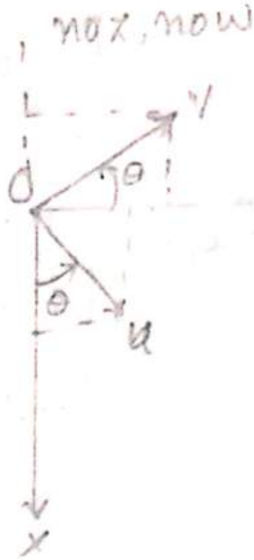


Rotation matrix abt Ox with angle, θ

$$R_{x,\theta} = \begin{bmatrix} i_x \cdot i_u & i_x \cdot j_v & i_x \cdot k_w \\ j_y \cdot i_u & j_y \cdot j_v & j_y \cdot k_w \\ k_z \cdot i_u & k_z \cdot j_v & k_z \cdot k_w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$\theta = 90°$.



$R_{x,\alpha}$; $R_{y,\phi}$; $R_{x,\theta}$ are called basic rotation matrices. Other finite rotation matrices can be obtained from these matrices.

① Given two points $a_{uvw} = (4,3,2)^T$ and $b_{uvw} = (6,2,4)^T$ w.r.t rotated OUVW coordinate s/m, determine the corr: points a_{xyz}, b_{xyz} w.r.t reference coordinate s/m if it has been rotated 60° abt the Oz axis.

$\cos 60 = 0.5$
$\sin 60 = 0.866$

$$a_{xyz} = R_{z,60}\, a_{uvw} \quad \text{and} \quad b_{xyz} = R_{x,60}\, b_{uvw}$$

$$a_{xyz} = \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(0.5)+3(-0.866)+2(0) \\ 4(0.866)+0.5(3)+0(2) \\ 0(4)+0(3)+1(2) \end{bmatrix}$$

$$= \begin{bmatrix} -0.598 \\ 4.964 \\ 2.0 \end{bmatrix}$$

$c\phi \equiv \cos\phi$; $s\phi \equiv \sin\phi$; $c\theta \equiv \cos\theta$; $s\theta \equiv \sin\theta$

$c\alpha \equiv \cos\alpha$; $s\alpha \equiv \sin\alpha$

If ϕ along OY
θ along OZ
α along OX

$$R = R_{x,\alpha} R_{z,\theta} R_{y,\phi}$$

$$R = \begin{bmatrix} c\theta c\phi & -s\theta & c\phi s\phi \\ c\alpha s\theta c\phi + s\alpha s\phi & c\alpha c\theta & c\alpha s\theta s\phi - s\alpha c\phi \\ s\alpha s\theta c\phi - c\alpha s\phi & s\alpha c\theta & s\alpha s\theta s\phi + c\alpha c\phi \end{bmatrix}$$

In addn, OUVW, rotating coordinate s/m can also rotate abt its own principal axes.

<u>Certain rules are to be obeyed</u>:-

1. Initially both coordinate s/ms are coincident, hence the rotation matrix is a 3x3 identity matrix, I_3
2. If the rotating coordinate s/m OUVW is rotating abt one of the principal axes of the OXYZ frame, then then premultiply the previous (resultant) rotation matrix ~~of the o~~ with an appropriate basic rotation matrix.
3. If the rotating coordinate s/m OUVW is rotating abt its own principal axes, then post multiply the previous (resultant) rotation matrix with an appropriate basic rotation matrix.

Eg:- rotation of ϕ angle abt OY axis followed by a rotation of θ angle abt OW axis followed by a rotation of α angle abt OU axis :

$$R = R_{y,\phi} I_3 R_{w,\theta} R_{u,\alpha} = R_{y,\phi} R_{w,\theta} R_{u,\alpha}$$

same as $R_{y,\phi} R_{z,\theta} R_{x,\alpha}$ seq: of rotations are different

eg:- $R_{x,\theta}\ R_{z,\theta}\ R_{z,\alpha}\ R_{y,\beta}\ R_{x,\delta}$

$R_{x,\theta} . R_{z,\alpha}\ R_{y,\beta}$

$R_{y,\phi} R_{z,\alpha}$
$x, u \to \alpha$
$y, v \to \phi$
$z, w \to \theta$

Geometric Interpretation of Rotation matrices

* fixed point p in OUVW coordinate s/m to be $(1 0 0)^T$

$P_{uvw} \equiv i_u = p_x i_u + p_y j_v + p_z k_w$. As $p_u = 1$

1^{st} column of rotation matrix represents coordinates of this pt. w.r.t OXYZ coordinate s/m.

* Given a reference frame OXYZ and a rotation matrix, column vectors of rotation matrix represent the principal axes of the OUVW coordinate s/m w.r.t ref: frame ⇒ we can draw the location of all the principal axes of the OUVW coordinate frame w.r.t reference frame (OXYZ)

* Similarly, inverse of a rotation matrix is equal to its transpose, row vectors of the rotation matrix represent the principal axes of the ref: s/m OXYZ w.r.t rotated coordinate s/m OUVW.

* provides insight into many robot arm kinematics problems.

Useful properties of Rotation matrices.

1. Each column vector of the rotation matrix is a repⁿ of rotated axis unit vector expressed in terms of the axis unit vectors of the reference frame,
Each row vector – repⁿ of axis unit vector of the reference frame. expressed in terms of the OUVW of rotated axis unit vectors of frame,

RM^T = row vectors of RM represent the P.A of reference frame w.r.t rotated coordinate s/m

2. Since each row and column is a unit vector repn magnitude of each row and column should be equal to

Direct property of orthonormal coordinate s/ms

|R| of rotation matrix = +1 for a rt hand coordinate s/m
" = −1 " left hand "

3. Since each row is a vector repn of orthonormal vectors dot pdt of each row with each other =0. & for column

4. Inverse of a rotation matrix = Transpose of the rotation matrix.

$$R^{-1} = R^T \quad \& \quad RR^T = I_3.$$

Properties 3 & 4 used to check the results of rotation matrix mulxns & in determining an erroneous row/column vector.

Homogeneous coordinates and Transformation Matrix

* Translation and Scaling is impossible to represent in a 3×3 rotation matrix.
A fourth coordinate / component is introduced to a position vector, $p = (p_x, p_y, p_z)^T$ in a 3D space which makes $\hat{p} = (\omega p_x, \omega p_y, \omega p_z, \omega)^T$
$\hat{p}$ is expressed in homogeneous coordinates.

* useful in developing matrix transformations that include rotation, translation, scaling & perspective transformation.

* N component position vector when represented by an (N+1) component vector is called <u>homogeneous coordinate transformation</u>.

* N dimensional vector is obtained by dividing homogeneous coordinates by the $(N+1)^{th}$ coordinate, w

ie. $p_x = \frac{wp_x}{w}$ $\quad p_y = \frac{wp_y}{w}$ $\quad p_z = \frac{wp_z}{w}$

* No unique homogeneous coordinates rep^n for a position vector in 3D space.

$\hat{p}_1 = (w_1 p_x, w_1 p_y, w_1 p_z, w_1)^T$; $\hat{p}_2 = (w_2 p_x, w_2 p_y, w_2 p_z, w_2)^T$

These are all homogeneous coordinates representing same position vector, $\bar{p} = (p_x, p_y, p_z)^T$ and "w" will be the scale factor. If $w = 1$, physical coordinates and homogeneous coordinate vectors are the same. [Robotics ⇒ $w = 1$]

* <u>Homogeneous coordinates are formed by 4 sub matrices</u>

$$T = \left[\begin{array}{c|c} R_{3\times3} & P_{3\times1} \\ \hline f_{1\times3} & 1\times1 \end{array}\right] = \left[\begin{array}{c|c} \text{rotation matrix} & \text{position vector} \\ \hline \text{perspective transformation} & \text{scaling} \end{array}\right]$$

$R_{x,\alpha} \longrightarrow T_{x,\alpha}$.

$$T_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad T_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

These are basic homogeneous rotation matrices

* $P_{3\times1}$ submatrix has the effect of translating OUVW coordinate s/m which has axes || to OXYZ & these origin is at (dx, dy, dz) of reference coordinate s/m.

$$T_{tran} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Basic homogeneous Translation matrix.

* $f_{1\times3}$ represents perspective transformation useful for computer vision and calibration of camera models. Presently elements are set to zero to indicate null perspective transformation.

* Diagonal elements are meant for local & global scaling. 1st 3 diagonal elements produce local stretching/scaling

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix}$$

* Basic rotation matrix Trot do not have any scaling effect. 4th diagonal element produces global scaling

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix}$$ where $s > 0$

$P_x = \frac{x}{s}$ $P_y = \frac{y}{s}$ $P_z = \frac{z}{s}$ $\omega = \frac{s}{s} = 1$

* Reduces the coordinates if $s > 1$ &
enlarges " if $0 < s < 1$

$\hat{P}_{xyz} = T \hat{P}_{uvw}$ with scaling factor : 1

$$T = \left[\begin{array}{ccc|c} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} n & s & a & p \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(0 → perspective, 1 → scaling; upper arrows: → 0, → P)

Composite homogeneous transformation matrix

Rules :-

1. Initially both coordinate systems are coincident, hence the homogeneous transformation matrix is a 4×4 identity matrix, I_4
2. If the rotating coordinate s/m OUVW is rotating/translating abt the principal axes of OXYZ, then premultiply previous homogeneous frame, then premultiply previous homogeneous transformation matrix with an appropriate basic homogeneous rot/translation matrix.
3. If the rotating coordinate s/m OUVW is rotating/translating abt its own principal axes, then post multiply the previous homogeneous transformation matrix with an appropriate basic homogeneous rot/translation matrix.

Eg: ① Two points $a_{uvw} = (4,3,2)^T$ and $b_{uvw} = (6,2,4)^T$ are to be translated a distance of +5 units along OX axis and −3 units along OZ axis. Determine a_{xyz} and b_{xyz}

$$\hat{a}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\hat{b}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Translated points are $a_{xyz} = (9, 3, -1)^T$

$b_{xyz} = (11, 2, 1)^T$

2. Find T that represents a rotation of α angle abt OX axis, followed by a translation of a units along
OX axis, " " d "
OZ axis, " " θ "
OZ axis.

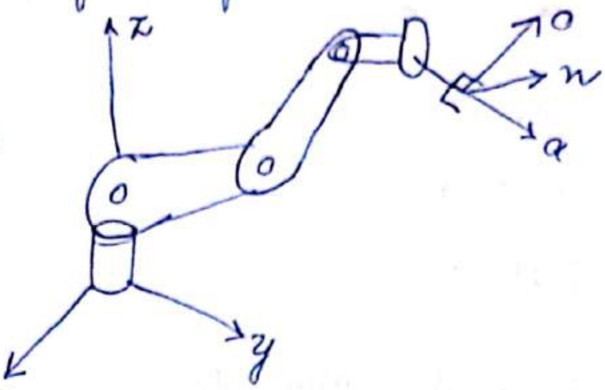
$$T = T_{z,\theta}\, T_{z,d}\, T_{x,a}\, T_{x,\alpha}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\cos\alpha \sin\theta & \sin\alpha \sin\theta & a\cos\theta \\ \sin\theta & \cos\alpha \cos\theta & -\sin\alpha \cos\theta & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Consider a fixed frame (Universal) F_{xyz} and set of axes n, o, a to represent moving frame F_{noa} relative to fixed frame.

n – normal
o – orientation
a – approach.

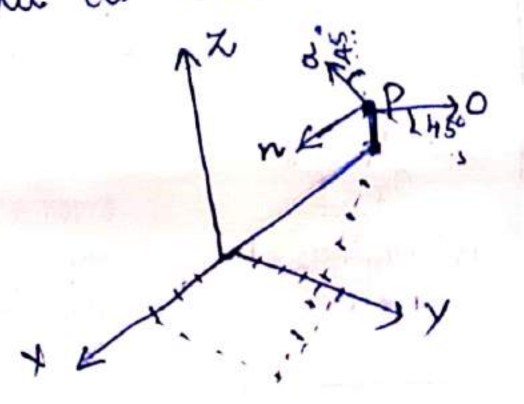


$$F = \begin{bmatrix} n_x & o_a & a_x \\ n_y & o_y & a_y \\ n_z & o_z & a_z \end{bmatrix}$$

When a frame, F is fully described both the location of its origin and the direction of its axes must be specified.

$$F = \begin{bmatrix} n_x & o_x & a_x & P_x \\ n_y & o_y & a_y & P_y \\ n_z & o_z & a_z & P_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(1) A frame F shown is located at 3, 5, 7 units with its n-axis ∥ to x, o-axis at 45° relative to the y-axis and its a-axis at 45° relative to the z-axis.



$$F = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 0.707 & -0.707 & 5 \\ 0 & 0.707 & 0.707 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Constraints translate into the following 6 constraint equations

1. $n \cdot o = 0$
2. $n \cdot a = 0$
3. $a \cdot o = 0$.

} n, o, a are mutually perpendicular. $n \times o = a$.

4. $|n| = 1$
5. $|o| = 1$
6. $|a| = 1$.

2. Find the values of the missing elements and complete the matrix.

$$F = \begin{bmatrix} n_x & o_x & a_x & 5 \\ n_y & o_y & a_y & 3 \\ n_z & o_z & a_z & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad F = \begin{bmatrix} ? & 0 & ? & 5 \\ 0.707 & ? & ? & 3 \\ ? & ? & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x o_x + n_y o_y + n_z o_z = 0$

$n_x a_x + n_y a_y + n_z a_z = 0$

$a_x o_x + a_y o_y + a_z o_z = 0$.

$n_x^2 + n_y^2 + n_z^2 = 1$

$o_x^2 + o_y^2 + o_z^2 = 1$

$a_x^2 + a_y^2 + a_z^2 = 1$

$n_x(0) + 0.707(o_y) + n_z o_z = 0$

$n_x(a_x) + 0.707(a_y) + n_z(0) = 0$

$a_x(0) + a_y(o_y) + 0(o_z) = 0$

$n_x^2 + 0.707^2 + n_z^2 = 1$

$0^2 + o_y^2 + o_z^2 = 1$

$a_x^2 + a_y^2 + 0^2 = 1$

3. $0.707 o_y + n_z o_z = 0$

5. $n_x a_x + 0.707 a_y = 0$

1. $a_y o_y = 0$

4. $n_x^2 + n_z^2 = 0.5$

2. $o_y^2 + o_z^2 = 1$

$a_x^2 + a_y^2 = 1$

1. $o_y = 0$. 4. $n_x = \pm 0.707$

2. $o_z = 1$

3. $n_z = 0$.

6. $a_y = -0.707$

$\pm 0.707 \times a_x = -0.707 a_y$.

5. $\pm 0.707 a_x = -0.707 \times -0.707$

$a_x = \frac{+0.707 \times 0.707}{\pm 0.707}$

$= \pm 0.707$

n_x and a_x = same sign

$$F = \begin{bmatrix} 0.707 & 0 & 0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$o_x^2 + o_y^2 + o_z^2 = 1$

$o_x = \underline{0}$

$a_x^2 + a_y^2 + a_z^2 = 1$

$1 + a_z^2 = 1$

$a_z^2 = 0$.

$n_x^2 + n_y^2 + n_z^2 = 1$

$0.707^2 + n_y^2 + 0^2 = 1$

$n_y^2 = 0.5$

$n_y \pm 0.707$

$$F_2 = \begin{bmatrix} -0.707 & 0 & -0.707 & 5 \\ 0.707 & 0 & -0.707 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

③ Find the missing elements of the foll: frame repn.

$$F = \begin{bmatrix} \overset{n_x}{-} & \overset{o_x}{0} & \overset{a_x}{-} & 3 \\ \overset{n_y}{0.5} & \overset{o_y}{-} & \overset{a_y}{-} & 9 \\ \overset{n_z}{0} & \overset{o_z}{-} & \overset{a_z}{-} & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$n_x(0) + 0.5(o_y) + 0(o_z) = 0$

$n_x(a_x) + 0.5(a_y) + 0(a_z) = 0$

$a_x(0) + a_y(o_y) + a_z(o_z) = 0$

$n_x^2 + 0.5^2 + 0 = 1$

$0^2 + o_y^2 + o_z^2 = 1$

$a_x^2 + a_y^2 + a_z^2 = 1$

$0.5\, o_y = 0$

$n_x a_x + 0.5 a_y = 0$

$a_y o_y + a_z o_z = 0$

$n_x^2 = 0.75$

$o_y^2 + o_z^2 = 1$

$a_x^2 + a_y^2 + a_z^2 = 1$

$n_x = \underline{\underline{0.866}}$

$o_y = \underline{\underline{0}}$

$o_z = \underline{\underline{1}}$

$a_y(0) + a_z(1) = 0$

$a_z = \underline{\underline{0}}$

$0.866(a_x) + 0.5 a_y = 0.$

$a_x = \underline{\underline{0.5}}$

$a_y = \underline{\underline{-0.866}}$

$$F = \begin{bmatrix} 0.866 & 0 & 0.5 & 3 \\ 0.5 & 0 & -0.866 & 9 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Homogeneous Transformation Matrices

1. Keep matrix in square (easy to find inverse of square matrix)
2. To multiply two matrices, dimensions must match

Representation, $F = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Pure Translation

$$T = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ no rotational movement.

→ translation.

If some movement is applied, $F_{new} = Trans(dx, dy, dz) \times F_{old}$.

④ A frame F has been moved 10 units along the y-axis and 5 units along the z-axis of the reference frame. Find new location of the frame.

$$F = \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{new} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 3 \\ -0.766 & 0 & 0.643 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.527 & -0.574 & 0.628 & 5 \\ 0.369 & 0.819 & 0.439 & 13 \\ -0.766 & 0 & 0.643 & 13 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pure Rotation

$$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$Rot(y,\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

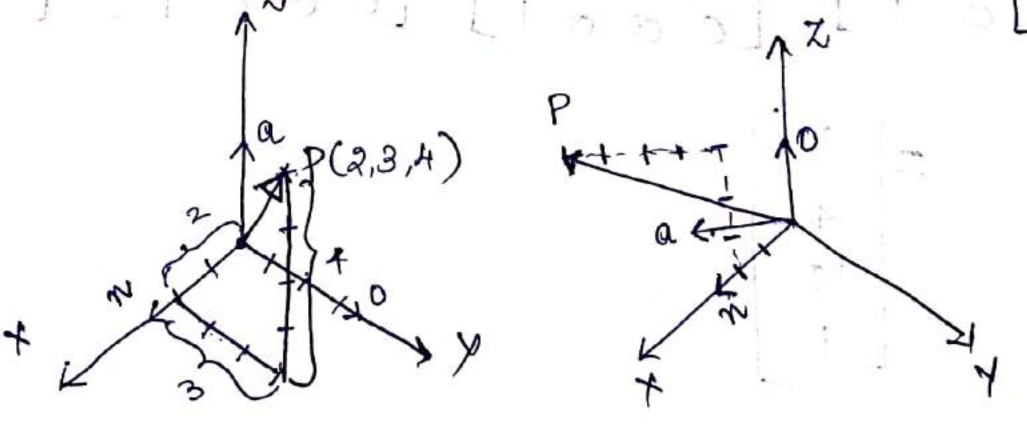
$$Rot(z,\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{xyz} = Rot(x,\theta) \times P_{noa}.$$

⑤ A point $P(2,3,4)^T$ is attached to a rotating frame. The frame rotates 90° about the x-axis of the reference frame. Find the co-ordinates of the point relative to the reference frame after the rotation and verify the result graphically.

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 90 & -\sin 90 \\ 0 & \sin 90 & \cos 90 \end{bmatrix} \times \begin{bmatrix} P_n \\ P_o \\ P_a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$



Combined Transformations

Assume frame, F_{noa} is subjected to the following 3 successive transformations relative to reference frame F_{xyz}

(a) Rotation of α degree about the x axis, $R(x, \alpha)$

(b) Followed by a translation of (l_1, l_2, l_3), $Trans(l_1, l_2, l_3)$)

(c) Followed by a rotation of β degrees about the y-axis, $R(y, \beta)$

$$P_{xyz} = R(y, \beta) \times Trans(l_1, l_2, l_3) \times R(x, \alpha) P_{noa}.$$

(6) A point $P(7,3,1)^T$ is attached to a frame F_{noa} and is subjected to the following transformations. Find the coordinates of the point relative to the reference frame at the conclusion of transformations.

1. Rotation of 90° about the z-axis
2. Followed by a rotation of 90° about the y-axis
3. Followed by a translation of [4, -3, 7]

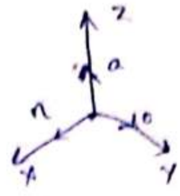
$$P_{xyz} = Trans(4, -3, 7) \times Rot(y, 90) \times Rot(z, 90) P_{noa}$$

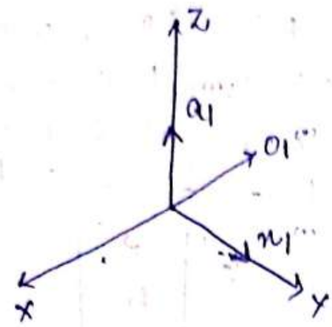
$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

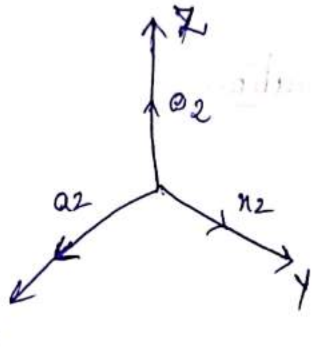
Solving Graphically

After the 1st Transformation Rot (x,90)





After the 2nd Transformation Rot (y,90)

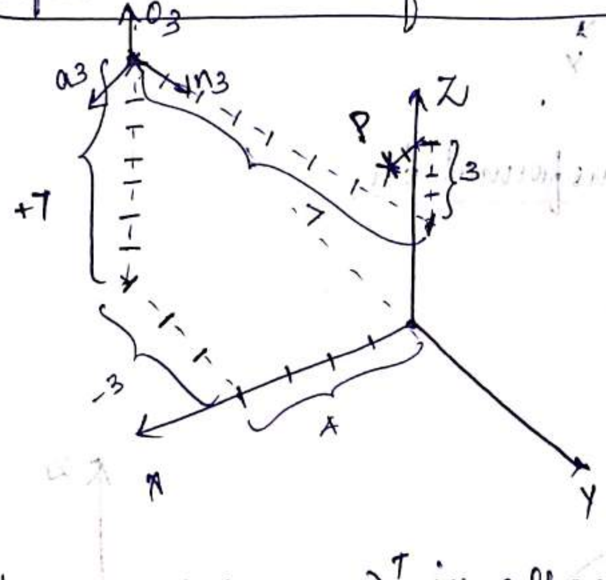


After the 3rd Transformation Trans (4,-3,7)

(a) x-axis = 4 + 1 = 5

(n) y-axis = -3 + 7 = 4

(o) z-axis = 7 + 3 = 10



7) Assume $P(7,3,1)^T$ is attached to F_{noa} is subjected to transformations in different order.

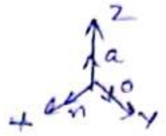
1. Rotation of 90° about the z-axis
2. Followed by a Translation of [4,-3,7]
3. Followed by a Rotation of 90° abt, the Y-axis.

$$P_{xyz} = Rot(y,90) \times Trans[4,-3,7] \times Rot(z,90) \times P_{noa}$$

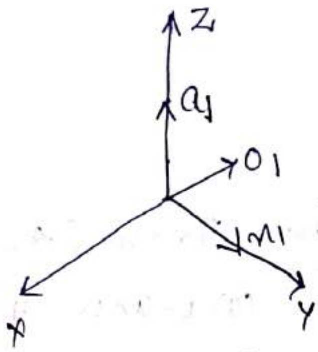
$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 4 \\ -1 \\ 1 \end{bmatrix}$$

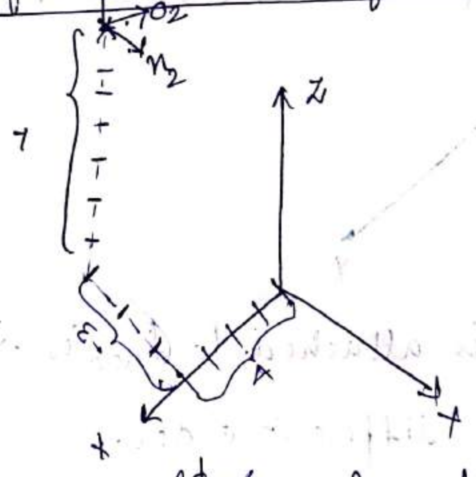
Solving Graphically



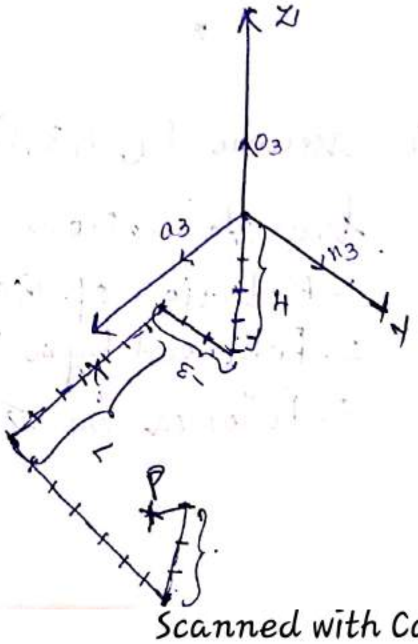
After the 1st Transformation



After the 2nd Transformation



After the 3rd Transformation



x-axis (a_3) = 7+1 = 8

y-axis (n_3) = -3+7 = 4

z-axis (o_3) = -4+3 = 1

(8) Assume the point $P(7,3,1)^T$ subjected to the transformations, relative to the moving frame. Find the co-ordinates of the point relative to the reference frame after transformations.

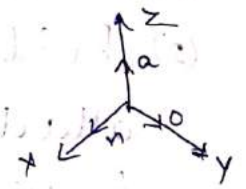
1. A rotation of 90° about the a-axis.
2. Then a translation of [4,-3,7] along n-o-a axes
3. Followed by a rotation of 90° about o-axis.

Here the transformations are made relative to the current frame, each transformation matrix is post-multiplied.

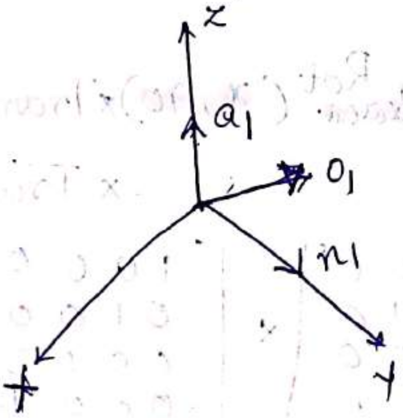
$$P_{xyz} = Rot(a,90) \times Trans(4,-3,7)\, Rot(o,90)\, P_{noa}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} x-n \\ y-o \\ z-a \\ \end{matrix} \times \begin{bmatrix} 7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

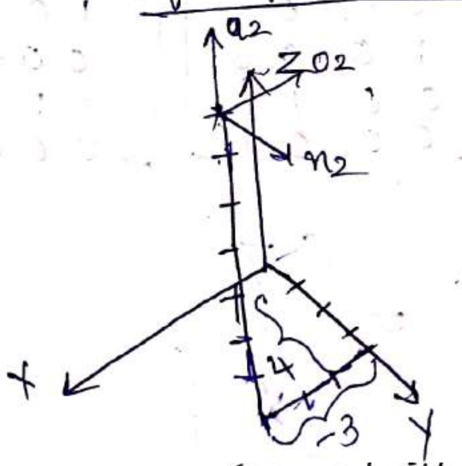
$$= \begin{bmatrix} 0 \\ 5 \\ 0 \\ 1 \end{bmatrix}$$



After 1st Transformation

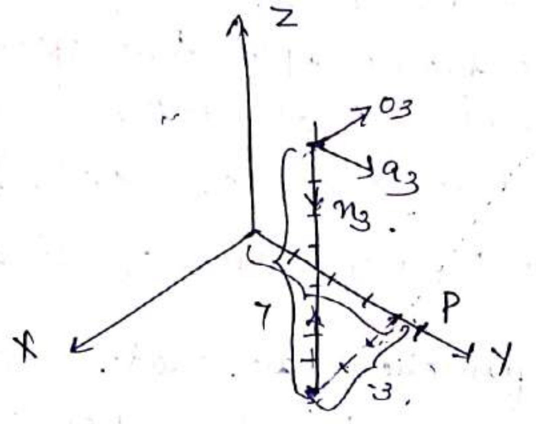


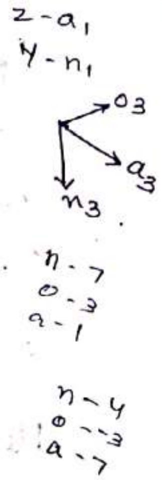
After 2nd Transformation



Scanned with CamScanner

After 3rd Transformation





~~x axis~~ $n_3 = 7 - 7 = \underline{0}$.

~~y axis~~ $a_3 = \underline{5}$.

~~z axis~~ $o_3 = -3 + 3 = \underline{0}$.

x axis (o_3)
y axis (a_3)
z axis (n_3)

(9) A frame B was rotated about the x-axis 90°, then it was translated about the current a axis 3" before it was rotated about the z-axis 90°. Finally it was translated about current o-axis 5".

(1) Write equation that describes motion.

(2) Find the final location of a point $P(1,5,4)^T$ attached to the frame relative to the reference frame.

(1) $T_B = Rot(z,90) \times Rot(x,90) \times Trans(0,0,3) \times Trans(0,5,0)$

(2) $$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 5 \\ 4 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 1 \\ 10 \\ 1 \end{bmatrix}$$

(10) A frame F was rotated about the y axis 90°, followed by a rotation about the o-axis of 30° followed by a translation of 5 units along n axis and finally a translation of 4 units along the x-axis.

T = Trans(4,0,0) × Rot(y,90) × Rot(o,30) × Trans(5,0,0)

$$= \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.866 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 & 0 & 0.866 & 1.5 \\ 0 & 1 & 0 & 0 \\ -0.866 & 0 & -0.5 & -4.33 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find inverse of Transformation Matrix

$$Rot(x,\theta)^{-1} = Rot(x,\theta)^T$$

ie. $$Rot(x,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

$$|Rot(x,\theta)| = 1\left[C^2\theta + S^2\theta\right] = 1$$

$$Rot(x,\theta)^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & S\theta \\ 0 & -S\theta & C\theta \end{bmatrix}$$

$$adj\left[Rot(x,\theta)\right] = Rot(x,\theta)^T$$

$Rot(x,\theta)^{-1} = Rot(x,\theta)^T$

A matrix with these characteristics are unitary matrix. All rotation matrices are unitary matrices. So to calculate inverse of a Rotation matrix is to transpose it.

For a homogenous 4×4 Transformation matrix, it is shown that the matrix inverse can be written by dividing the matrix into two portions:

(1) Rotation portion of matrix is simply transposed as it is unitary.

(2) Position portion of 4×4 matrix is the negative of the dot product of the p-vector with each of the n-, o-, a- vectors.

Eg:- $T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $T^{-1} = \begin{bmatrix} n_x & n_y & n_z & -p.n \\ o_x & o_y & o_z & -p.o \\ a_x & a_y & a_z & -p.a \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11) Calculate the matrix representing $Rot(x,40)^{-1}$

$$Rot(x,40) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & -0.643 & 0 \\ 0 & 0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$Rot(x,40)^{-1} = Rot(x,40)^T$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.766 & 0.643 & 0 \\ 0 & -0.643 & 0.766 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(12) Calculate the inverse of the given Transformation Matrix

$$T = \begin{bmatrix} 0.5 & 0 & 0.866 & 3 \\ 0.866 & 0 & -0.5 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T^{-1} = \begin{bmatrix} 0.5 & 0.866 & 0 & -(3\times0.5+2\times0.866+5\times0) \\ 0 & 0 & 1 & -(3\times0+2\times0+5\times1) \\ 0.866 & -0.5 & 0 & -(3\times0.866+2\times-0.5+5\times0) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.866 & 0 & -3.23 \\ 0 & 0 & 1 & -5 \\ 0.866 & -0.5 & 0 & -1.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orientation

There are three common configurations.

(1) Roll, Pitch, Yaw (RPY) angles.

(2) Euler angles.

(3) Articulated joints.

RPY angles

Rotation of ϕ_a about a axis (x-axis of moving frame) - Roll

" ϕ_o about o axis (Y axis ") - Pitch

" ϕ_n " n axis (X axis ") - Yaw

$$RPY(\phi_a, \phi_o, \phi_n) = Rot(a, \phi_a)\, Rot(o, \phi_o)\, Rot(n, \phi_n)$$

Euler angles.

Rotation of ϕ about a-axis (x-axis of moving frame)
followed by,
Rotation of θ about o-axis (Y axis ")
followed by,
Rotation of ψ about a axis (z axis ")

$$\text{Euler}(\phi, \theta, \psi) = \text{Rot}(a,\phi)\,\text{Rot}(o,\theta)\,\text{Rot}(a,\psi)$$

$$= \begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta & 0 \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse Kinematic solution for the Euler angles can be found in a manner similar to RPY.

Premultiply euler eqn by $Rot^{-1}(a,\phi)$ on two sides to eliminate ϕ from one side.

$$Rot^{-1}(a,\phi) \times \begin{bmatrix} n_x & o_x & a_x & 0 \\ n_y & o_y & a_y & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\theta C\psi & -C\theta S\psi & S\theta & 0 \\ S\psi & C\psi & 0 & 0 \\ -S\theta C\psi & S\theta S\psi & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (A)$$

or

$$\begin{bmatrix} n_x C\phi + n_y S\phi & o_x C\phi + o_y S\phi & a_x C\phi + a_y S\phi & 0 \\ -n_x S\phi + n_y C\phi & -o_x S\phi + o_y C\phi & -a_x S\phi + a_y C\phi & 0 \\ n_z & o_z & a_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (B)$$

(1) Comparing (A) & (B)

$-a_x S\phi + a_y C\phi = 0$. So $\phi = \text{atan2}(a_y, a_x)$

(2) $S\psi = -n_x S\phi + n_y C\phi$
$C\psi = -o_x S\phi + o_y C\phi$

$\psi = \text{atan2}\left[(-n_x S\phi + n_y C\phi), (-o_x S\phi + o_y C\phi)\right]$

(3) $S\theta = a_x C\phi + a_y S\phi$
$C\theta = a_z$.

$\theta = \text{atan2}\left[(a_x C\phi + a_y S\phi), a_z\right]$

(13) Find necessary euler angle for the given Transformation matrix

$$\begin{bmatrix} n_x & o_x & a_x & p_x \\ 0.579 & -0.548 & -0.604 & 5 \\ 0.540 & 0.813 & -0.220 & 7 \\ 0.611 & -0.199 & 0.766 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\phi = \text{atan2}(a_y, a_x) = \text{atan2}(-0.22, -0.604)$
$= 20^\circ$ or 200°.

$\psi = \text{atan2}(0.31, 0.952) = 18^\circ$ or 198°

$\theta = \text{atan2}(-0.643, 0.766) = -40^\circ$ or 40°.