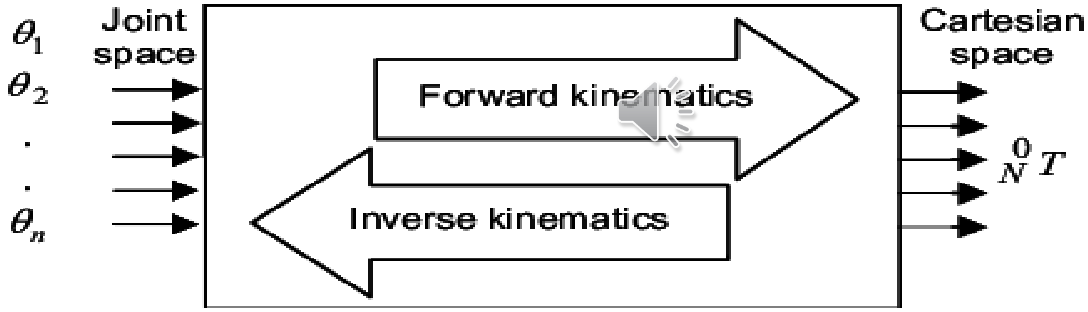
MODULE-4-INVERSE KINEMATICS

EC 368-ROBOTICS

What is Forward Kinematics and Inverse Kinematics?



Problem Formulation

- Forward Kinematics
  - End effector position as a function of q
  - We solve for $H_n^0(q)$.

- Inverse Kinematics
  - Given $H = T_n^0(q_1, \dots, q_n) = A_1(q_1) \cdots A_n(q_n)$
  - Find all joint variables $q_1, \dots, q_n$
    - Can rewrite as 12 equations in n unknowns as $T_{ij}(q_1, \dots, q_n) = h_{ij}$
    - Why not 16 equations?

- Last row is always considered as [0 0 0 1],neglect that row then only 12 simultaneous equations are needed for solution.

METHODS OF SOLUTION

- We have to find joint variables in terms of the end effector position and orientation which is more difficult.
- There are two ways :(1)Algebraic approach

  (2) Geometrical approach
- In Algebraic method,end effector Position parameters(x,y and z) is obtained from Homogenous Transformation matrix.
- By applying Cosine rules
- Pythagoras theorem
- Sum of angles identities
- Use atan2 (arc tangent function)

It is used in applications involving vectors in Euclidean space ,such as finding the direction from one point to another. To convert Rotation matrix representation into Euler angle.

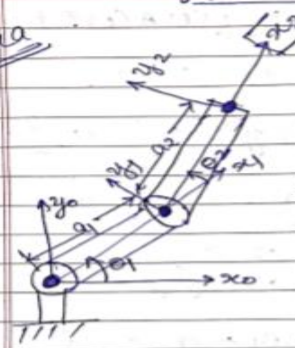
atan2(y,x)=Cartesian to polar

Gets value of y and x and assumes a complex number as x+iy and returns its phase

EG:Inverse Kinematics of RR Planar Elbow Manipulator-Algebraic Approach

To find Inverse Kinematics

using algebra



From Homogenous Transformation matrix

Step1: Take out dx & dy values alone.

$$T_2^0 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \rightarrow x \\ \rightarrow y \\ \\ \\ \end{matrix}$$

$c_{12} = \cos(\theta_1 + \theta_2)$

$s_{12} = \sin(\theta_1 + \theta_2)$

$x = a_1 \cos\theta_1 + a_2 \cos(\theta_1 + \theta_2)$

$y = a_1 \sin\theta_1 + a_2 \sin(\theta_1 + \theta_2)$

Continued...

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

Step 2 : Squaring and adding.

$$\Rightarrow x^2 + y^2 = \underbrace{\left\{ a_1^2 \cos^2\theta_1 + a_2^2 \cos^2(\theta_1+\theta_2) + 2a_1a_2 \cos\theta_1 \cdot \cos(\theta_1+\theta_2) \right\}}_{x^2} + \underbrace{\left\{ a_1^2 \sin^2\theta_1 + a_2^2 \sin^2(\theta_1+\theta_2) + 2a_1a_2 \sin\theta_1 \sin(\theta_1+\theta_2) \right\}}_{y^2}$$

Let $\theta_1 = A$, $\theta_1+\theta_2 = B$.

$$\left[\cos\theta_1 \cdot \cos(\theta_1+\theta_2) + \sin\theta_1 \sin(\theta_1+\theta_2) = \cos A \cos B + \sin A \sin B = \cos(A-B) \right]$$

So & $\cos^2\theta + \sin^2\theta = 1$

$$\Rightarrow a_1^2 + a_2^2 + 2a_1a_2 \cos[\theta_1 - \theta_1 - \theta_2]$$

$$x^2+y^2 \Rightarrow a_1^2 + a_2^2 + 2a_1a_2 \cos\theta_2 \rightarrow (1)$$

$$\cos\theta_2 = \frac{x^2 + y^2 - a_1^2 - a_2^2}{2a_1a_2}$$

$$\theta_2 = \cos^{-1}\left[\frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}\right] \Rightarrow (A)$$

Continued….

Step 3: Apply sum of angles identities & substitute $\cos\theta_2 \rightarrow c_2$, $\sin\theta_2 \rightarrow s_2$

$$\begin{cases} x = (a_1 + a_2 c_2)\cos\theta_1 - a_2 s_2 \sin\theta_1 \\ y = (a_1 + a_2 c_2)\sin\theta_1 + a_2 s_2 \cos\theta_1 \end{cases}$$

Obtained from:

$$x = a_1 c_1 + a_2(c_1 s_2 - s_1 s_2)$$

$$= (a_1 + a_2 c_2)c_1 - a_2 s_2 s_1$$

$$y = a_1 s_1 + a_2 \quad (s_1 c_1 + c_1 s_1)$$

$$= (a_1 + a_2 c_1)s_1 + a_2 s_2 c_1$$

Consider y.

$$y = (a_1 + a_2 c_2)\sin\theta_1 + a_2 s_2 \cos\theta_1 \rightarrow (1)$$

Note:

$$a\cos\theta + b\sin\theta = c$$

$$\theta = \tan^{-1}\frac{c}{\sqrt{a^2+b^2-c^2}} - \tan^{-1}\left(\frac{a}{b}\right)$$

Continued....

From (2)

$$a = a_2 s_2 \; ; \; b = a_1 + a_2 c_2 \; ; \; c = y$$

$$\theta_1 = \tan^{-1} \frac{y}{\sqrt{a_2^2 s_2^2 + a_1^2 + a_2^2 c_2^2 + 2 a_1 a_2 c_2 - y^2}} - \tan^{-1}\left(\frac{a_2 s_2}{a_1 + a_2 c_2}\right)$$

$$\theta_1 = \tan^{-1}\left(\frac{y}{\text{sqrt}\, a_1^2 + a_2^2 + 2 a_1 a_2 c_2 - y^2}\right) - \tan^{-1}\left(\frac{a_2 s_2}{a_1 + a_2 c_2}\right)$$

From (1) $x^2 + y^2 = a_1^2 + a_2^2 + 2 a_1 a_2 c_2$

$$\theta_1 = \tan^{-1}\left(\frac{y}{\text{sqrt}\, x^2 + y^2 - y^2}\right) - \tan^{-1}\left(\frac{a_2 s_2}{a_1 + a_2 c_2}\right)$$

$$\boxed{\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{a_2 s_2}{a_1 + a_2 c_2}\right)} \Rightarrow (B)$$

Joint variables of RR (2 link planar elbow manipulator) θ_1 and θ_2 is thus found out

Redundant solution:

Redundant solutions are available while performing Inverse Kinematics

$$\cos\theta_2 = \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2}$$

$$\sin\theta_2 = \pm\sqrt{1-\cos^2\theta_2}$$

So θ_2 cannot be confirmed as $\cos^{-1}$

There comes the importance of atan2

$$\theta_2 = \operatorname{atan2}(\sin\theta_2, \cos\theta_2)$$

$$\boxed{\theta_2 = \operatorname{atan2}(s_2, c_2)}$$

Condition for this solution to exist:

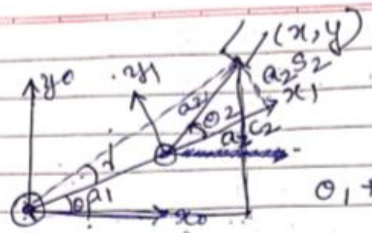
if $-1 \leq c_2 \leq 1$

$$-1 \leq \frac{x^2+y^2-a_1^2-a_2^2}{2a_1a_2} \leq 1$$

Rewriting

$$(a_1-a_2)^2 \leq x^2+y^2 \leq (a_1+a_2)^2$$

Continued....



$\theta_1 + \gamma =$

$\sin\theta_2 = \frac{opp}{a_2}$; $opp = a_2 s_2$

$\cos\theta_2 = \frac{adj}{a_2}$; $adj = a_2 c_2$

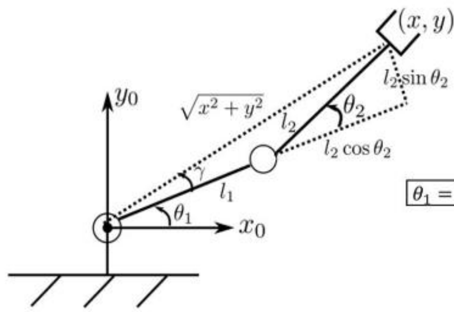
$\tan(\theta_1 + \gamma) = \frac{y}{x}$

$\theta_1 + \gamma = atan2(y, x)$

$\gamma = atan2(a_2 s_2, a_1 + a_2 c_2)$

$$\theta_1 = atan2(y, x) - atan2(a_2 s_2, a_1 + a_2 c_2)$$

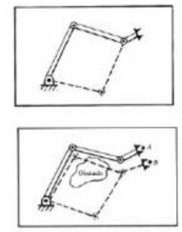
Solve for θ_1



$\theta_1 + \gamma = \text{atan2}(y, x)$

$\gamma = \text{atan2}(l_2 s_2, l_1 + l_2 c_2)$

$$\boxed{\theta_1 = \text{atan2}(y, x) - \text{atan2}(l_2 s_2, l_1 + l_2 c_2)}$$

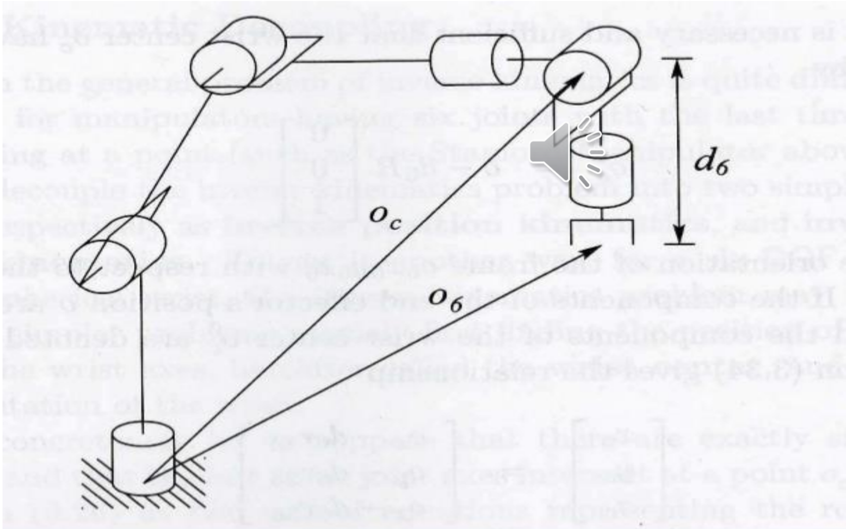


2 solutions:
elbow up,
elbow down

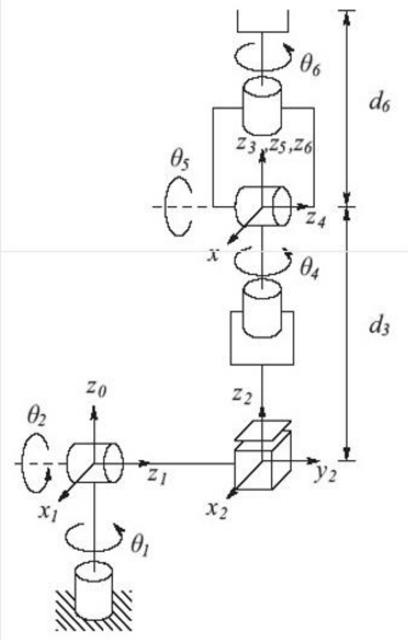
KINEMATIC DECOUPLING

- Some complicated problems like Stanford manipulator and for manipulators with 6 DOF with last 3 joint axes intersect at a point (wrist centre),general IK is difficult So Kinematic decoupling is done.
- For that Decouple into two solutions: inverse position of wrist centre

  inverse orientation of wrist centre



Kinematic Decoupling



Introduction Robotics, lecture 2 of 7

link	a_i	α_i	d_i	θ_i
1	0	−90	0	θ_1^*
2	0	90	d_2	θ_2^*
3	0	0	d_3^*	0
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

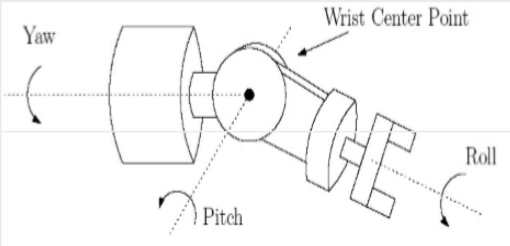
$$A_i = \begin{bmatrix} c_{\theta_i} & -s_{\theta_i}c_{\alpha_i} & s_{\theta_i}s_{\alpha_i} & a_i c_{\theta_i} \\ s_{\theta_i} & c_{\theta_i}c_{\alpha_i} & -c_{\theta_i}s_{\alpha_i} & a_i s_{\theta_i} \\ 0 & s_{\alpha_i} & c_{\alpha_i} & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

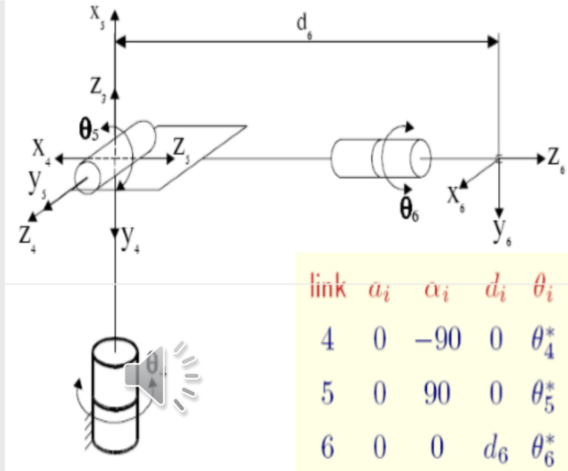
- IK problem: for given R and o solve 9 rotational and 3 positional equations:

$$R_6^0(q_1, q_2, \ldots, q_6) = R$$

$$o_6^0(q_1, q_2, \ldots, q_6) = o$$

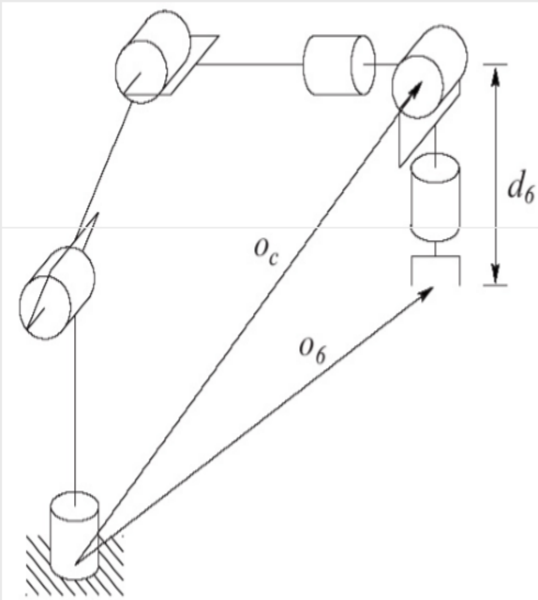
- Spherical wrist as paradigm.





link	a_i	α_i	d_i	θ_i
4	0	−90	0	θ_4^*
5	0	90	0	θ_5^*
6	0	0	d_6	θ_6^*

- Let o_c be the intersection of the last 3 joint axes; as z_3, z_4, and z_5 intersect at o_c, the origins o_4 and o_5 will always be at o_c;
  the motion of joints 4, 5 and 6 will not change the position of o_c;
  only motions of joints 1, 2 and 3 can influence position of o_c.



Introduction Robotics, lecture 3 of 7

$$o = o_c^0 + d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$o = \begin{bmatrix} o_x \\ o_y \\ o_z \end{bmatrix} \quad o_c^0 = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$$

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} o_x - d_6 r_{13} \\ o_y - d_6 r_{23} \\ o_z - d_6 r_{33} \end{bmatrix} \Rightarrow q_1, q_2, q_3$$

$$R = R_3^0 R_6^3 \quad \Rightarrow$$

$$R_6^3 = (R_3^0)^{-1} R = (R_3^0)^T R \Rightarrow q_4, q_5, q_6$$