May 2019 (KTU)

Part C.

14 (b). Consider a linear system described by the transfer function $\frac{Y(s)}{U(s)} = \frac{10}{s(s+1)(s+2)}$.
Design a feedback controller with a state feedback so that the closed loop poles are placed at $-2, -1 \pm 1j$

Solution.

Step 1. $\frac{Y(s)}{U(s)} = \frac{10}{s(s+1)(s+2)}$

Crossmultiplying

$$Y(s)\,(s)(s+1)(s+2) = 10\,U(s).$$

$$Y(s)\left[s^3 + 3s^2 + 2s\right] = 10\,U(s).$$

$$s^3 Y(s) + 3s^2 Y(s) + 2s\,Y(s) = 10\,U(s)$$

Taking inverse LT

$$\dddot{y} + 3\ddot{y} + 2\dot{y} = 10u.$$

Put $\dddot{y} = \dot{x}_3$; $x_1 = y$; $x_2 = \dot{y}$; $x_3 = \ddot{y}$

$$\dot{x}_3 + 3x_3 + 2x_2 = 10u.$$

$$\dot{x}_3 = -2x_2 - 3x_3 + 10u.$$

$$\dot{x}_2 = x_3.$$

$$\dot{x}_1 = x_2.$$

The state model in matrix form;

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} [u].$$

Check for controllability.

Find Q_c; $Q_c = [B \;\; AB \;\; A^2B]$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 6 & 7 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 10 \\ -30 \\ 70 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \\ -30 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 0 & 10 \\ 0 & 10 & -30 \\ 10 & -30 & 70 \end{bmatrix}$$

$$|Q_c| = \begin{vmatrix} 0 & 0 & 10 \\ 0 & 10 & -30 \\ 10 & -30 & 70 \end{vmatrix} = -1000.$$

$$\neq 0.$$

System is completely state controllable.

Step - 2

To determine the characteristic polynomial;

desired ~~ploied~~ closed loop poles are

$\mu_1 = -2$; $\mu_2 = -1+1j$ $\mu_3 = -1-1j$

Desired characteristic polynomial is

$$(\lambda-\mu_1)(\lambda-\mu_2)(\lambda-\mu_3) = 0.$$

$$= (\lambda+2)(\lambda+1-j)(\lambda+1-j) = 0.$$

$$(\lambda+2)((\lambda+1)^2-(1j^2)) = 0.$$

$$(\lambda+2)(\lambda^2+2\lambda+1+1) = 0.$$

$$(\lambda+2)(\lambda^2+2\lambda+2) = 0$$

$$\lambda^3+4\lambda^2+6\lambda+4 = 0.$$

Desired characteristic polynomial is

$$\lambda^3+4\lambda^2+6\lambda+4 = 0. \quad \text{—— (1)}$$

Step - 3

To determine the gain matrix k.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix}$$

Let $k = [k_1, k_2, k_3]$

Characteristic polynomial with state feedback is

$$|\lambda I - (A-BK)| = 0.$$

$$[\lambda I - A + BK] = \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 10k_1 & 10k_2 & 10k_3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10k_1 & 2+10k_2 & \lambda+3+10k_3 \end{bmatrix}$$

$$|\lambda I - A + BK| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 10k_1 & 2+10k_2 & \lambda+3+10k_3 \end{vmatrix}$$

$$\lambda(\lambda(\lambda+3+10k_3) + 2 + 10k_2) + 10k_1 = 0.$$

$$\lambda^3 + (3+10k_3)\lambda^2 + (2+10k_2)\lambda + 10k_1 = 0$$

Step 4.

Equating the coefficients characteristic polynomial ① and ②

$10k_1 = 4$. ; $k_1 = \frac{4}{10} = 0.4$.

$2 + 10k_2 = 6$. ; $10k_2 = 4$; $k_2 = \frac{4}{10} = 0.4$.

$3 + 10k_3 = 4$ $10k_3 = 1$; $k_3 = \frac{1}{10} = 0.1$

$$k = [k_1 \; k_2 \; k_3] = [0.4 \;\; 0.4 \;\; 0.1]$$

State feedback gain matrix.

$$k = [0.4 \;\; 0.4 \;\; 0.1]$$

KTU – April 2018

Part C – 14 (a).

Qn Consider a system defined by

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u \text{ and } y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.$$

Using state feedback control $u = -kx$, it is desired to have the closed loop poles at $s = -3$; $s = -4$, determine the state feedback gain matrix k.

Solution:

$$A = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix};$$

(From the block diagram (theory) of state feedback controller, 'C' has no effect on the feedback controller).

Step 1

Check for controllability.

$$Q_c = [B \quad AB]$$

$$AB = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} 0 & 2 \\ 2 & 2 \end{bmatrix}$$

$$|Q_c| = -4 \neq 0.$$

The system is completely controllable.

Step 2.

To determine the characteristic polynomial desired closed loop poles are

$$\mu_1 = -3 \; ; \; \mu_2 = -4 .$$

Desired characteristic polynomial is

$$(\lambda - \mu_1)(\lambda - \mu_2) = 0 .$$

$$(\lambda+3)(\lambda+4) = 0 .$$

$$\lambda^2 + 7\lambda + 12 = 0 . \quad \text{—— (1)}$$

Step 3.

To determine gain matrix k;

Characteristic polynomial with state feedback is $|\lambda I - (A - Bk)| = 0$; $|\lambda I - A + Bk| = 0$.

$$\lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = 0 .$$

$$= \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2k_1 & 2k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda & -1 \\ 3+2k_1 & \lambda - 1 + 2k_2 \end{bmatrix}$$

$$|\lambda I - A + Bk| = \lambda(\lambda - 1 + 2k_2) + 3 + 2k_1 = 0 .$$

$$\lambda^2 - \lambda + 2k_2\lambda + 3 + 2k_1 = 0 .$$

$$\lambda^2 + \lambda(2k_2 - 1) + 3 + 2k_1 = 0 \text{ — (1)} .$$

Equating coefficients of 1 and 2;

$$7 = 2k_2 - 1$$

$$8 = 2k_2 \qquad k_2 = \cancel{k_2}\, 4$$

$\cancel{2+}\ 3 + 2k_1 = 12.$

$2k_1 = 9.$

$k_1 = \frac{9}{2} = 4.5.$

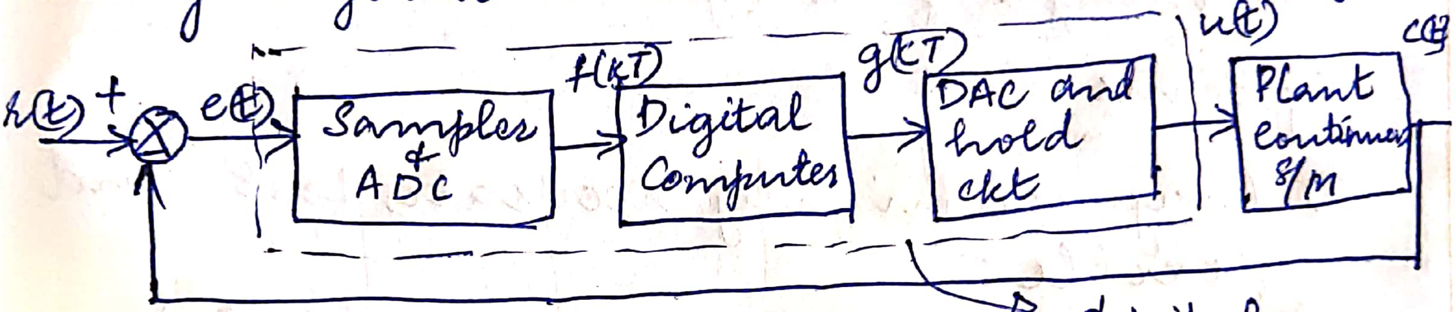
State feedback gain matrix

$$K = [k_1 \ \ k_2] = \underline{[4.5 \ \ 4]}$$

Sampled Data control systems:

When the signal or information in a system is in the form of discrete pulses, then the system is called discrete data system. In control engineering; discrete data system is known as sampled data system.

A sampled data control system using digital controller is shown in fig.



The input and output signal in a digital computer will be digital signals. The error signal (i/p. to digital controller) and control signal to drive the plant (u(t)) are analog (continuous) in nature. Hence a sampler and analog to digital (ADC) are provided in the computer i/p. A digital to analog converter (DAC) and a hold ckt is provided at computer output.

The sampler converts the continuous time-error signal into a sequence of pulses and ADC produces the binary code for each sample. These codes are the input data to digital computer which process the binary codes and produces another stream of binary codes as output. The DAC and hold circuit converts the output binary codes to continuous time signal

<u>Advantages of sampled data control systems</u>

(1) They are highly accurate, fast and flexible.

(2) Time sharing concept of digital computers result in economical cost and space.

(3) Digital transducers used in the system have better resolution.

(4) The digital components are less affected by noise, nonlinearities and transmission errors of noisy channel.

5) Sampled data system require low power instruments which can be built to have high sensitivity.

6. The system performance can be modified by compensation techniques.

Z transform.

The laplace transform is used for analysis of continuous time signals. Similary Z transform is used in analysis and representation of linear discrete time systems.

Defenition of Z transform

Let $f(K)$ = Discrete time signal or sequence.

$$F(z) = Z\{f(k)\}.$$

$$F(z) = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$ where z is a complex variable.

The above equation is said to be two sided transform; and transform is two sided transform (k is defined for both +ve and -ve values).

One sided Z transform is defind as:

$$F(z) = Z\{f(k)\}.$$

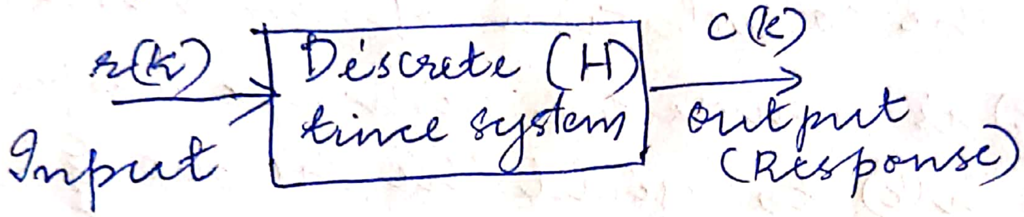
$$= \sum_{k=0}^{\infty} f(k) z^{-k}.$$

Some common one sided z transforms

$f(t)$	Function $f(k)(k>0)$	Z transform $F(z)$
	$\delta(k)$	1
	$u(k)$ or 1	$z(z-1)$
	a^k	$z(z-a)$
	$k a^k$	$\frac{a\,z}{(z-a)^2}$
	$k^2 a^k$	$\frac{a\,z(z+a)}{(z-a)^3}$
	$(k+1)a^k$	$\frac{z^2}{(z-a)^2}$
t	kT	$\frac{Tz}{(z-1)^2}$
t^2	$(kT)^2$	$\frac{T^2 z(z+1)}{(z-1)^3}$
e^{-at}	e^{-akT}	$\frac{z}{z-e^{-aT}}$
te^{-at}	kTe^{-akT}	$\frac{zTe^{-at}}{(z-e^{-aT})^2}$
$\sin \omega t$	$\sin \omega kT$	$\frac{z \sin \omega T}{z^2-2z\cos \omega T+1}$
$\cos \omega t$	$\cos \omega kT$	$\frac{z(z-\cos \omega T)}{z^2-2z\cos \omega T+1}$

Linear discrete time system (LDS)

A discrete time system is a device ~~Hard~~ or algorithm that operates on a discrete time signal called the input or excitation according to some well defined rule to produce another discrete time signal called the output or response of the system.



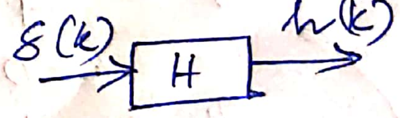
$$c(k) = H\,r(k)$$

A discrete time system is linear if it obeys the principle of superposition and it is time invariant if it's input-output relationship do not change with time.

Transfer function of LDS system (Pulse transfer function)

When the input to a discrete system is unit impulse, $\delta(k)$, then the output is called impulse response of the system and denoted by $h(k)$

$$h(k) = H(\delta(k))$$



Z transform of $h(k) = Z\{h(k)\} = H(z)$

$\therefore$ transfer function of LDS system $= H(z)$

The input output relationship of an LDS system is governed by a convolution sum.

$$c(k) = \sum_{m=0}^{\infty} r(m)\, h(k-m) \quad \text{— (1)}$$

where m is the summation of impulses.

Eqn (1) is the convolution of impulses.

Input $r(k)$ is convoluted with the impulse response $h(k)$ to yield the output $c(k)$

$$c(k) = \sum_{m=0}^{\infty} r(m)\, h(k-m) = r(k) * h(k)$$

Proof.

By the defenition of one sided transform

$$C(z) = Z\{c(k)\} = \sum_{k=0}^{\infty} c(k)\, z^{-k} \quad \text{— (2)}$$

But $c(k) = \sum_{m=0}^{\infty} r(m)\, h(k-m)$

Substituting the convolution sum in (2)

$$\cancel{C(z) = \sum_{k=0}^{m} c(k)}$$

$$C(z) = \sum_{k=0}^{\infty} \left[\sum_{m=0}^{\infty} r(m)\, h(k-m) \right] z^{-k} \quad \text{— (3)}$$

Order of summation can be interchang

$$C(z) = \sum_{m=0}^{\infty} \left[r(m) \sum_{k=0}^{\infty} h(k-m) \right] z^{-k} \quad \text{— (4)}$$

Let $p = k - m$ when $k = 0$; $p = -m$

$k = \alpha$; $p = \alpha$.

$k = p + m$.

Substituting for $(k-m)$ and k in ④

$$C(z) = \sum_{m=0}^{\alpha} r(m) \sum_{p=-m}^{\alpha} h(p) z^{-(p+m)}$$

$$= \sum_{m=0}^{\alpha} r(m) \sum_{p=0}^{\alpha} h(p) z^{-p} z^{-m} \cdot \quad (h(p)=0 \text{ for } p<0.$$

$$= \sum_{m=0}^{\alpha} r(m) z^{-m} \sum_{p=0}^{\infty} h(z) z^{-p}$$

By the defenitiop of one sided Z transform.

$$\sum_{m=0}^{\alpha} r(m) z^{-m} = R(z)$$

$$\sum_{p=0}^{\alpha} h(p) z^{-p} = H(z)$$

$$\therefore \quad C(z) = R(z) H(z).$$

or $$\boxed{H(z) = \frac{C(z)}{R(z)}}$$

Thus transfer function of an LDS system is the ratio of Z transform of the output of a system to the z transform of the input to the system with zero initial conditions.

Pulse transfer function $H(z) = \dfrac{C(z)}{R(z)}$