MODULE 4

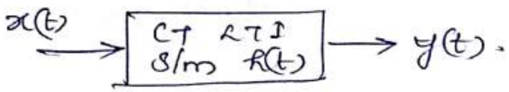
I Analysis of LTI System using Fourier transform

II. Analysis of LTI System using Laplace transform.

III. Sampling

I. Analysis of LTI system using Fourier transform

Consider a continous time LTI system with impulse response $h(t)$ and input $x(t)$. $y(t)$ is the corresponding system output.



The output of CT LTI system can be found by using convolution operation

$$y(t) = x(t) * h(t).$$

Now Taking fourier transform on both sides.

$$F[x(t) * h(t)] = X(\omega) \cdot H(\omega)$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

Here $H(\omega)$ is the transfer function of continous time LTI system.

The transfer function of LTI system in frequency domain is equal to the ratio of fourier transform of output function to the fourier transform of input signal.

Now the impulse response $h(t) = F^{-1}[H(\omega)]$
Impulse response can be found by taking the inverse fourier transform of $H(\omega)$.

Here $H(\omega)$ is the complex function of ω
It can be expressed in terms of magnitude function and phase function

$$H(\omega) = |H(\omega)| \angle H(\omega)$$

where $|H(\omega)| = \sqrt{(H_I(\omega))^2 + (H_R(\omega))^2}$

$$|\angle H(\omega)| = \tan^{-1}\left[\frac{H_I(\omega)}{H_R(\omega)}\right]$$

The frequency response of continuous time LTI system is same as the transfer function in frequency domain.

$H(\omega)$ = frequency response / Transfer function

Q.1→ The input and output of a causal LTI system are related by the differential equation

$$\frac{d^2y(t)}{dt^2} + 6\cdot\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

(a) find the IR of the system

(b) what is the response of this system if $x(t) = te^{-2t}\cdot u(t)$.

Soln

The constant coefficient differential equation is given by

$$\frac{d^2y(t)}{dt^2} + 6\cdot\frac{dy(t)}{dt} + 8y(t) = 2x(t)$$

Now applying FT on both sides

$$F[y(t)] = Y(\omega)$$

$$F\left[\frac{dy(t)}{dt}\right] = j\omega\, Y(\omega)$$

$$F\left(\frac{d^2y(t)}{dt^2}\right] = (j\omega)^2\cdot Y(\omega)$$

$$(j\omega)^2 Y(\omega) + 6\cdot j\omega\cdot Y(\omega) + 8Y(\omega) = 2X(\omega)$$

$$Y(\omega)\left[(j\omega)^2 + 6j\omega + 8\right] = 2X(\omega)$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{(j\omega)^2 + 6j\omega + 8} = \frac{2}{(j\omega+4)(j\omega+2)}$$

$$\frac{2}{(j\omega+4)(j\omega+2)} = \frac{A}{j\omega+4} + \frac{B}{j\omega+2} \quad \text{——— (1)}$$

$$\frac{2}{\cancel{(j\omega+4)}\cancel{(j\omega+2)}} = \frac{A(j\omega+2) + B(j\omega+4)}{\cancel{(j\omega+4)}\cancel{(j\omega+2)}}$$

$$2 = A(j\omega+2) + B(j\omega+4)$$

Put $j\omega = -2$

$$2 = B(2)$$

$$\underline{\underline{B = 1}}$$

put $j\omega = -4$

$$2 = A(-2)$$

$$A = -1$$

Put the value of $A \& B$ in ①

$$H(\omega) = \frac{-1}{j\omega+4} + \frac{1}{j\omega+2}$$

$$h(t) = F^{-1}[H(\omega)] = F^{-1}\left[\frac{-1}{j\omega+4}\right] + F^{-1}\left[\frac{1}{j\omega+2}\right]$$

$$\underline{\underline{h(t) = -e^{-4t}\cdot u(t) + e^{-2t}\cdot u(t)}}$$

(II) $x(t) = t e^{-2t}\cdot u(t).$

$$X(\omega) = \frac{1}{(j\omega+2)^2}$$

$$Y(\omega) = X(\omega)\cdot H(\omega)$$

$$= \frac{1}{(j\omega+2)^2}\cdot\frac{2}{(j\omega+4)\cdot(j\omega+2)}$$

$$= \frac{2}{(j\omega+2)^3(j\omega+4)}$$

$$F[e^{-2t}\cdot u(t)] = \frac{1}{j\omega+2}$$

$$F[t\cdot e^{-2t}\cdot u(t)] = ?$$

$$j\cdot\frac{d}{d\omega}X(\omega)$$

$$= j\cdot\frac{d}{d\omega}\left[\frac{1}{j\omega+2}\right]$$

$$= j\left[\frac{(j\omega+2)0 - 1.(j)}{(j\omega+2)^2}\right]$$

$$= \frac{1}{(j\omega+2)^2}$$

$$\frac{2}{(j\omega+2)^3(j\omega+4)} = \frac{A}{j\omega+2} + \frac{B}{(j\omega+2)^2} + \frac{C}{(j\omega+3)^3} + \frac{D}{(j\omega+4)}$$

By solving we get

$$A = \frac{1}{4},\ B = \frac{-1}{2},\ C = 1,\ D = \frac{-1}{4}$$

$$Y(\omega) = \frac{1}{4(j\omega+2)} - \frac{1}{2}\cdot\frac{1}{(j\omega+2)^2} + \frac{1}{(j\omega+2)^3} - \frac{1}{4(j\omega+2}$$

$$= \frac{1}{4}\cdot e^{-2t}\cdot u(t) - \frac{1}{2}\, t\cdot e^{-2t}\cdot u(t) + \frac{t^2}{2}\cdot e^{-2t}\cdot u(t) - \frac{1}{4}\cdot e^{-4t}$$

Q.2. Find the frequency response of an LTI system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2x(t)$$

Soln: To find $H(j\omega)$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

Given: $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = 2x(t)$

Apply FT on both sides

$$(j\omega)^2\cdot Y(j\omega) + 5\cdot j\omega\cdot Y(j\omega) + 6Y(j\omega) = 2\cdot X(j\omega)$$

$$Y(j\omega)\left[(j\omega)^2 + 5j\omega + 6\right] = 2X(j\omega)$$

$$\frac{Y(j\omega)}{X(j\omega)} = \frac{\cancel{(j\omega)^2+5j\omega+6}\ 2}{(j\omega)^2+5j\omega+6}$$

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II. Analysis of LTI Systems using Laplace transform

Let us consider a continuous time LTI system with input $x(t)$ and impulse response $h(t)$. The output $y(t)$ of the system can be found by using convolution

$$x(t) \longrightarrow \boxed{\text{CT LTI} \; h(t)} \longrightarrow y(t)$$

$$y(t) = x(t) * h(t)$$

Applying Laplace transform on both sides

$$Y(s) = X(s) \cdot H(s)$$

$$H(s) = \frac{Y(s)}{X(s)}$$

- Impulse response $h(t) = L^{-1} H(s)$

- Step response. $y(t) = L^{-1}[Y(s)]$

$$= L^{-1}[X(s) \cdot H(s)]$$

$$= L^{-1}\left[\frac{1}{s} \cdot H(s)\right]$$

The total response of the system can be expressed as the sum of two components

Total response = Natural response + Forced response

The natural response of the system is due to initial conditions of the s/m. While the forced response is due to input alone.

Here we need to know

$$\mathcal{L}[y(t)] = Y(s)$$

$$\mathcal{L}\left[\frac{dy(t)}{dt}\right] = sY(s) - y(0)$$

$$\mathcal{L}\left[\frac{d^2y(t)}{dt^2}\right] = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}\left(\frac{d^3y(t)}{dt^2}\right] = s^3Y(s) - s^2y(0) - sy'(0) - y''(0)$$

Q.) A system is described by the foll differential equation

$$\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 12y(t) = x(t).$$

Determine the response of the system to a unit step input applied at $t=0$. The initial conditions are $y(0) = \cancel{0}\,-2$ & $\frac{dy(0)}{dt} = 0$

Soln

Here input $x(t) = u(t)$ and initial conditions are also given

$$\frac{d^2y(t)}{dt^2} + 7 \cdot \frac{dy(t)}{dt} + 12\, y(t) = x(t)$$

Applying Laplace transform on both sides

$$s^2Y(s) - s\,y(0) - y'(0) + 7\,(s\,Y(s) - y(0)] + 12\,Y(s) = X(s)$$

Here $y(0) = -2$ &
$y'(0) = 0$
$x(t) = u(t)$
$X(s) = 1/s$

Applying conditions, we get

$$s^2Y(s) - s(-2) - 0 + 7\,[s\,Y(s) - (-2)] + 12\,Y(s) = 1/s$$

$$s^2Y(s) + 2s + 7s\,Y(s) + 14 + 12\,Y(s) = 1/s$$

$$Y(s)\,[\,s^2 + 7s + 12] = 1/s - 2s - 14$$

$$Y(s)\,[s^2 + 7s + 12] = \frac{1 - 2s^2 - 14s}{s}$$

$$Y(s) = \frac{1 - 2s^2 - 14s}{s\,(s+3)(s+4)} = A/s + \frac{B}{s+3} + \frac{C}{s+4} \quad \text{--- (1)}$$

By solving we get $A = 1/12$, $B = \frac{-25}{3}$, $C = \frac{25}{4}$

By substituting these values in (1)

$$Y(s) = \frac{1}{12.s} + \frac{-25}{3} \times \frac{1}{s+3} + \frac{25}{4} \cdot \frac{1}{s+4}$$

$$y(t) = L^{-1}[Y(s)] = \frac{1}{12}(1) - \frac{25}{3}\,e^{-3t} u(t) + \frac{25}{4}\,e^{-4t} \cdot u(t)$$

Q. For a system with transfer function

$$H(s) = \frac{s+5}{s^2+5s+6}$$

find the zero state response with the input $x(t) = e^{-3t} u(t)$.

Soln: Zero state response means the response of the s/m due to zero initial conditions and by applying input alone

$$H(s) = \frac{s+5}{s^2+5s+6}$$

$$\frac{Y(s)}{X(s)} = \frac{s+5}{(s+3)(s+2)}$$

Here the given input is $x(t) = e^{-3t} u(t)$

$$\therefore X(s) = \frac{1}{s+3}$$

Substitute $1/s+3$ in place of $X(s)$

$$\frac{Y(s)}{1/s+3} = \frac{s+5}{(s+3)\cdot(s+2)}$$

$$Y(s)\cdot(s+3) = \frac{s+5}{(s+3)\,s+2} = \frac{s+5}{(s+3)^2(s+2)}$$

$$Y(s) = \frac{s+5}{(s+3)^2(s+2)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s+2}$$

By solving we get $C = 3$, $A = -1$

$B = -2$

$$Y(s) = \frac{-1}{S+3} - \frac{2}{(S+3)^2} + \frac{3}{S+2}$$

$$y(t) = \mathcal{L}^{-1}[Y(s)] = -e^{-3t} \cdot u(t) - 2te^{-3t} \cdot u(t) + 3e^{-2t} \cdot u(t)$$

Q. Find the impulse and step response of the following systems

1) $H(S) = \frac{10}{S^2+6S+10}$

Impulse response $h(t) = \mathcal{L}^{-1}[H(S)]$

$$= \mathcal{L}^{-1}\left[\frac{10}{(S+3)^2+1^2}\right]$$

$$= 10 e^{-3t} \cdot \sin t \; u(t)$$

Step input. $x(t) = u(t)$

$X(S) = 1/s$

$$\frac{Y(S)}{X(S)} = \frac{10}{S^2+6S+10}$$

$$\frac{Y(S)}{1/s} = \frac{10}{S^2+6S+10}$$

$$Y(S) = \frac{10}{S \cdot S^2+6S+10} = \frac{A}{S} + \frac{BS+C}{S^2+6S+10}$$

$A = 1, \; B = -1, \; C = -6$

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$$Y(s) = 1/s - \frac{s-6}{s^2+6s+10}$$

$$= 1/s - \frac{(s+6)}{s^2+6s+10}$$

$$= 1/s - \frac{(s+6)}{(s+3)^2+1}$$

$$= 1/s - \left(\frac{(s+3)}{(s+3)^2+1} + \frac{3}{(s+3)^2+1} \right]$$

$$y(t) = \left[1 - e^{-3t} \cdot \cos t - 3 e^{-3t} \cdot \sin t \right] u(t)$$

III SAMPLING

The sampling is a process of converting continous time signal in to discrete time signal. The sampling is done by taking samples of continous time signal at definite interval of time. If the time interval between two successive samples will be same and such type of sampling is called uniform sampling

The time interval between two samplesg is called sampling interval and is denoted by T. The inverse of sampling interval is called sampling frequency.