Module - V

Manipulator Dynamics - Intro to Lagrangian Mechanics and Dynamic eqn for 2 DOF robots, Intro to posn & force control of robotic manipulators, Robot actuation & control using PID, ~~Digital & PLC~~

Velocity Kinematics - Derivation of the Jacobian, Application of Velocity Kinematics for serial manipulators, Importance of Singularities.

No: of hours assigned - 6 ; Total hrs taken - 7hrs.

* Velocity kinematics.
* derive velocity relationships, relating linear and angular velocities of the end-effector to the joint variables. (velocities)
* determined by the Jacobian of function b/w space of cartesian positions and space of joint position (f/w kinematic eqns)
* Jacobian is a matrix - ordinary derivative of a scalar function.
* important quantity in the analysis and ctrl of robot motion.

Angular Velocity

$\omega = \dot{\theta} k$ where $\dot{\theta} = \frac{d\theta}{dt}$ (θ → angle) ; k - unit vector in the dirn of axis of rotation.

$v = \omega \times r$ where r - vector from the origin

→ linear velocity.

* Role of angular velocity – induce linear velocities of points in a rigid body.

Skew Symmetric Matrices.

1. $S^T + S = 0$ → ①, S is said to be skew symmetric

2. If set of 3×3 matrix, S_{ij} $i, j = 1,2,3$

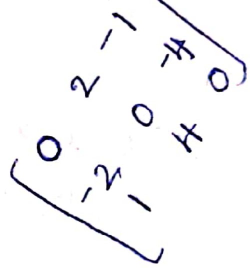
$$S_{ij} + S_{ji} = 0 \quad i,j = 1,2,3 . \rightarrow ②$$

$S_{ii} = 0$ diagonal ~~no~~ terms of S are zero.

$S_{ij}, i \neq j \quad S_{ij} = -S_{ji}$

$$S = \begin{bmatrix} 0 & -S_3 & S_2 \\ S_3 & 0 & -S_1 \\ -S_2 & S_1 & 0 \end{bmatrix}$$

(S_{11}, S_{12}, S_{13}; S_{21}, S_{22}, S_{23}; S_{31}, S_{32}, S_{33})



① $i = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Denote by i, j, k. the three unit basis coordinate vectors

$$S(i) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \quad S(j) = \begin{bmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

$$S(k) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Properties of Skew Symmetric Matrices.

a & b - vectors, α & β - scalar.

1. S is linear ie. $S(\alpha a + \beta b) = \alpha S(a) + \beta S(b)$ → ③

2. $S(a)p = a \times p$ → ② a & p - vectors -
(vector cross pdt)

3. $R S(a) R^T = S(Ra)$ → ③

$R(a \times b) = Ra \times Rb$. iff R is orthogonal.

Proof

$R\,S(a)\,R^T b = R(a \times R^T b)$
$= R(a) \times R R^T b$
$= R(a) \times b = S(Ra) \times b$.

4. $X^T S X = 0$ → ④

Derivation of Jacobian Matrix

Consider an n-link manipulator with joint variables $q_1 \ldots q_n$.

$$T_n^0(q) = \begin{bmatrix} R_n^0(q) & O_n^0(q) \\ 0 & 1 \end{bmatrix}$$ → Transformation from end-effector to base frame.

$q = [q_1 \ldots q_n]^T$ is the vector of joint variables.

Relate linear & angular velocity of the end-effector to the vector of joint velocities, $\dot{q}(t)$

$J = 6 \times n$ matrix
n = no: of links

$S(\omega_n^0) = \dot{R}_n^0 (R_n^0)^T$; $v_n^0 = \dot{O}_n^0$

↓ angular velocity of end-effector

↓ linear velocity

$$\left.\begin{aligned} v_n^0 &= J_v \cdot \dot{q} \\ \omega_n^0 &= J_\omega \cdot \dot{q} \end{aligned}\right\} \xi = J\dot{q}$$

J → manipulator Jacobian / Jacobian.

ξ → body velocity vector.

Jacobian of any manipulator -

(a) Angular velocity.

def: ang: velocity of the end-effector relative to the base by expressing the angular velocity contributing by each joint in the orientation of the base frame & then summing

* i^{th} joint - revolute. $q_i = \theta_i$ & z_{i-1} axis of rotation.

ω_i^{i-1} - ang: velocity of link i relative to frame $O_{i-1} x_{i-1} y_{i-1} z_{i-1}$

$$\omega_i^{i-1} = \dot{q}_i z_i^{i-1} = \dot{q}_i k$$

k = unit coordinate vector $[0 \; 0 \; 1]^T$

* i^{th} joint - prismatic. Motion is translation & $\omega_i^{i-1} = 0$

$q_i = d_i$

Overall angular velocity of end-effector, ω_n^0 in the base frame;

$$\omega_n^0 = \rho_1 \dot{q}_1 k + \rho_2 \dot{q}_2 R_1^0 k + \dots + \rho_n \dot{q}_n R_{n-1}^0 k$$

$$\omega_n^0 = \sum_{i=1}^{n} \rho_i \dot{q}_i z_{i-1}^0$$

in which $\rho_i = 1$ if joint i is revolute and 0 if joint i is prismatic, since.

$$z_{i-1}^0 = R_{i-1}^0 k.$$

$$z_0^0 = k = [0 \; 0 \; 1]^T$$

$$J_\omega = [\rho_1 z_0, \rho_2 z_1 \dots \rho_n z_{n-1}]$$

(Lower half of Jacobian)

(b) Linear velocity ($\dot{o}_n^0$)

$$\dot{o}_n^0 = \sum_{i=1}^{n} \frac{\partial o_n^0}{\partial q_i} \dot{q}_i$$ → orientation of base frame. (By chain rule for differentiation)

ith column of Jv

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i}$$

Becomes linear velocity If $\dot{q}_i = 1$ & $\dot{q}_j = 0$.

* ith column of Jacobian can be generated by holding all joints fixed but the ith and actuating the ith at unit velocity.

Case 1: Prismatic joints

Fix all joints except a single prismatic joint

∴ joint i is prismatic, pure translation to the end effector

Direction of translation is || to z_{i-1} and magnitude is

d_i – DH joint variable for prismatic joint, i

Orientation of base frame, $\dot{o}_n^0 = \dot{d}_i R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \dot{d}_i z_{i-1}^0$

$$\boxed{J_{v_i} = z_{i-1}}$$

$$J_{v_i} = \frac{\partial o_n^0}{\partial q_i} = \frac{\partial}{\partial d_i} d_i z_{i-1}^0$$
$$= z_{i-1}$$

Case 2: Revolute joints

Fix all joints except a single revolute joints.

$q_i = \theta_i$

Linear velocity → $\omega \times r$ where $\omega = \dot{\theta}_i z_{i-1}$

$r = o_n - o_{i-1}$

$$\boxed{J_{v_i} = z_{i-1} \times (o_n - o_{i-1})}$$

Combining Linear and Angular velocity Jacobians

Linear

$J_v = [J_{v_1} \ldots J_{v_n}]$ Upper half of Jacobian.

$$J_{v_i} = \begin{cases} z_{i-1} \times (o_n - o_{i-1}), & \text{if } i^{th} \text{ joint is revolute} \\ z_{i-1}, & \text{" prismatic.} \end{cases}$$

angular

$J_w = [J_{w_1} \ldots J_{w_n}]$ Lower half of Jacobian.

$$J_{w_i} = \begin{cases} z_{i-1}, & \text{if } i^{th} \text{ joint is revolute} \\ 0, & \text{" prismatic} \end{cases}$$

Proof

Case 1 : Prismatic joints

$$\begin{bmatrix} R_n^0 & o_n^0 \\ 0 & 1 \end{bmatrix} = T_n^0 = T_{i-1}^0 \cdot T_i^{i-1} \cdot T_n^i$$

$$= \begin{bmatrix} R_{i-1}^0 & o_{i-1}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_i^{i-1} & o_i^{i-1} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_n^i & o_n^i \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_i^0 & R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_n^i & o_n^i \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} R_n^0 & R_i^0 o_n^i + R_{i-1}^0 o_i^{i-1} + o_{i-1}^0 \\ 0 & 1 \end{bmatrix}$$

i.e. $O_n^0 = R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1} + O_{i-1}^0$

Assume only "i" is moved then O_n^i & O_{i-1}^0 are constants.

∵ joint is prismatic, R_{i-1}^0 = constant.

From DH convention. $O_i^{i-1} = \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix}$.

$q_i = d_i$ (prismatic)

$$\frac{\partial}{\partial q_i} O_n^0 = \frac{\partial}{\partial d_i} R_{i-1} O_i^{i-1}$$

$$= R_{i-1}^0 \frac{\partial}{\partial d_i} \begin{bmatrix} a_i c_i \\ a_i s_i \\ d_i \end{bmatrix} = \not{d_i}\, R_{i-1}^0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \not{d_i}\, z_{i-1}^0$$

i.e. $J_{vi} = z_{i-1}$

Case 2: Revolute joints

$$\frac{\partial}{\partial \theta_i} O_n^0 = \frac{\partial}{\partial \theta_i} \left[R_i^0 O_n^i + R_{i-1}^0 O_i^{i-1} \right]$$

$$= \frac{\partial}{\partial \theta_i} (R_i^0 O_n^i) + \frac{\partial}{\partial \theta_i} (R_{i-1}^0 O_i^{i-1})$$

$\left\{ \frac{\partial}{\partial \theta} R_{k,\theta} = S(k) \cdot R_{k,\theta} \right\}$ → Note.

⇒ $S(k) \cdot R_i^0 O_n^i + S(k) \cdot R_{i-1}^0 O_i^{i-1}$

$\{ R(\theta) \cdot R(\theta)^T = 1 \}$ Assume $R_{i-1}^0 (R_{i-1}^0)^T = 1$ → Note.

⇒

$\Rightarrow R_{i-1}^{0}\, S(K) \cdot (R_{i-1}^{0})^{T} \cdot \underbrace{R_i^{0}\, O_n^{i}}_{b} + R_{i-1}^{0}\, S(K) (R_{i-1}^{0})^{T} R_{i-1}^{0}\, O_i^{i-1}$

(R, S(a), R T, b)

$\{R\, S(a) R^{T} b\} = S(Ra) \cdot b\} \rightarrow$ Note $\{R\, S(a) R^{T} = S(Ra)\}$,

$\Rightarrow S(R_{i-1}^{0} \cdot k) \cdot R_i^{0}\, O_n^{i} + S(R_{i-1}^{0} \cdot k) \cdot R_{i-1}^{0}\, O_i^{i-1}$

$\{R_{i-1}^{0} k = z_{i-1}^{0}\} \rightarrow$ Note.

$\Rightarrow S(z_{i-1}^{0}) R_i^{0}\, O_n^{i} + S(z_{i-1}^{0}) R_{i-1}^{0}\, O_i^{i-1}$

$\Rightarrow S(z_{i-1}^{0}) [R_i^{0}\, O_n^{i} + R_{i-1}^{0}\, O_i^{i-1}]$

As the effect of end effector in overall is translation.

$S(\underset{a}{z_{i-1}^{0}}) [\underset{p}{O_n^{0} - O_{i-1}^{0}}]$

$\{O_n^{i} + O_i^{i-1} = O_n^{i-1} ; \Rightarrow O_n^{0} - O_{i-1}^{0}\} \rightarrow$ Note.

$\{S(a) \cdot p = a \times p\} \rightarrow$ Note.

$\Rightarrow z_{i-1}^{0} \times (O_n^{0} - O_{i-1})$

linear $J_{v_i} = \begin{cases} z_{i-1} \times (O_n - O_{i-1}) & \rightarrow \text{for revolute joint } i \\ z_{i-1} & \rightarrow \text{" prismatic joint } i \end{cases}$

angular $J_{\omega_i} = \begin{cases} z_{i-1} & \rightarrow \text{for revolute joint } i \\ 0 & \rightarrow \text{" prismatic joint } i \end{cases}$

Problems

1. Jacobian of 2-link planar manipulator

Jacobian for revolute joint.

$$J_i = \begin{bmatrix} z_{i-1} \times (d_n - d_{i-1}) \\ z_{i-1} \end{bmatrix} \begin{matrix} \text{linear} \\ \text{ang} \end{matrix}$$

Draw from page 80 & 81 fig: & table. $\begin{bmatrix} J_v \\ J_\omega \end{bmatrix}$

NB: d or 0

Jacobian for prismatic joint

$$J_i = \begin{bmatrix} z_{i-1} \\ 0 \end{bmatrix} \begin{matrix} \text{linear} \\ \text{ang} \end{matrix}$$

where

$$z_{i-1} \times (d_n - d_{i-1}) = \frac{d}{d\theta_i}(d_n - d_{i-1})$$

$n = 2$

$i = 1$

$J = [J_1 \quad J_2]$

$$J_0 = \begin{bmatrix} \overset{J_1}{z_0 \times (d_2 - d_0)} & \overset{J_2}{z_1 \times (d_2 - d_1)} \\ z_0 & z_1 \end{bmatrix}$$

$$T_1^0 = A_1 = \begin{bmatrix} c_1 & -s_1 & 0 & a_1 c_1 \\ s_1 & c_1 & 0 & a_1 s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (d_1)$$

$$T_2^0 = A_1 \cdot A_2 = \begin{bmatrix} c_{12} & -s_{12} & 0 & a_1 c_1 + a_2 c_{12} \\ s_{12} & c_{12} & 0 & a_1 s_1 + a_2 s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (d_2)$$

$$d_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad d_1 = \begin{bmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{bmatrix} \quad d_2 = \begin{bmatrix} a_1 c_1 + a_2 c_{12} \\ a_1 s_1 + a_2 s_{12} \\ 0 \end{bmatrix}$$

$$z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad z_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

* $Z_0 \times (d_2 - d_0) = \frac{d}{d\theta_1}(d_2 - d_0)$

* $z_1 \times (d_2 - d_1) = \frac{d}{d\theta_2}(d_2 - d_1)$

* $$J = \begin{bmatrix} \frac{d}{d\theta}\begin{bmatrix} a_1 c_1 + a_2 c_{12} - 0 \\ a_1 s_1 + a_2 s_{12} - 0 \\ 0 - 0 \end{bmatrix} & \frac{d}{d\theta}\begin{bmatrix} a_1 c_1 + a_2 c_{12} - a_1 c_1 \\ a_1 s_1 + a_2 s_{12} - a_1 s_1 \\ 0 - 0 \end{bmatrix} \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -a_1 s_1 - a_2 s_{12} & -a_2 s_{12} \\ a_1 c_1 + a_2 c_{12} & a_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

Singularities

* 6 x n Jacobian matrix is a transformation matrix relating joint velocity to cartesian velocity.

[If matrix is non singular, then $\dot{q} = J^{-1} \dot{x}$
$|J| \neq 0$ Rank of $J = n$.]

($|J| = 0$ singular rank $< n$)

* $\dot{x} = J(q)\,\dot{q}$ $\qquad$ $\dot{x} = (v, \omega)^T \rightarrow$ end effector velocities

$dx = J(q) \cdot dq$ $\qquad$ $\dot{q}$ = joint velocities.

* Since Jacobian is a fn of q, the configuration for which rank of J decreases is of special significance.

* Singular configuration is that configuration for which rank of J ↓

* Singular configuration in which all joint velocities cannot be determined from the end-effector velocity
* certain directions of motion may be unattainable.
* correspond to points in the manipulator workspace that may be unreachable.
* Near singularities there'll not exist a unique soln to inverse kinematics problem. There may be no soln or infinitely many solutions.

Two types of singularities.

① Arm Singularity – resulting from motion of arm
② Wrist " – " wrist.

Manipulator Dynamics

* During a work cycle, a manipulator must accelerate move at constant speed and decelerate. This time varying position and orientation of the manipulator is termed its dynamic behaviour.
* Mathematical eqn which describes the dynamic behaviour of manipulator – manipulator dynamics. and these are a set of equations of motion (EOM)
* provides relationship b/w joint actuator torques and motion of links. Lagrangian – diff: b/w PE & KE

To find the manipulator eqns of motion, Lagrange-Euler (LE) approach. (energy based approach)

Newton-Euler (NE) approach (force-balance based approach) can be used.

Dynamic model based on LE formulation is obtained from the Lagrangian,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i \text{ or } F_i \quad i = 1, 2 \ldots n$$

Lagrange fn, $L = T - U$

where T – kinetic energy – depends on both posn & velocity of arm.

U – potential energy – " only on arm position

F_i - generalized force acting on the i^{th} joint.

In robotics,

* consists of a system of n-second order non linear differential eqn in the vector of joint variables q.

Force Control

* important control scheme reqd for tasks such as assembly, grinding etc which involves extensive contact with environment. Such tasks are handled by controlling the forces of interaction b/w the manipulator and the environment directly.

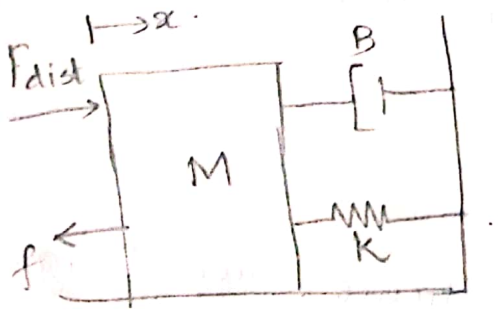
* A force feedback control algorithm accepts force & motion commands, measures forces and positions and produce motion commands to the manipulator

There are three types of sensors for force feedback

(a) Wrist force Sensor.
(b) Joint " torque sensors
(c) Tactile / hand sensors.

Wrist force sensors consist of an array of strain gauges located on the wrist.
Joint torque sensors consist of strain gauges on the actuator shaft.
Tactile sensors are located on the fingers of the grippers and are used for sensing gripping force and for shape detection.

Force control along a single DOF



Mass - Spring damper s/m.

Consider a system consisting of an end effector contacting the environment along the direction labeled x.
The environment is labeled as a 2nd order s/m consisting of inertia Me, stiffness ke and damping ratio Be.
Me - inertia of the end-effector + inertia of any tool in the gripper etc.

Eqn of the s/m is

$$M_e \ddot{x} + B_e \dot{x} + K_e x = F - F_{dist} \rightarrow ①$$

F – input force exerted by the end effector

F_{dist} – disturbance force.

$\dot{x} = \frac{\dot{F}_e}{M_e}$ $\ddot{x} = \frac{\ddot{F}_e}{M_e}$

Force exerted on the environment, $F_e = K_e x$; $K_e \dot{x} = \dot{F}_e$; $K_e \ddot{x} = \ddot{F}_e$

Eqn ① written in terms of Fe (Multiply eqn ① $\times \frac{K_e}{M_e}$)

$$\ddot{F}_e + \frac{B_e}{M_e} \dot{F}_e + \frac{K_e}{M_e} F_e = \frac{K_e}{M_e}(F - F_{dist})$$

$e_f = F_{ed} - F_e$

If the desired force trajectory $F_{ed}(t)$ is given, then the feed forward control scheme is analogous to posn control scheme.

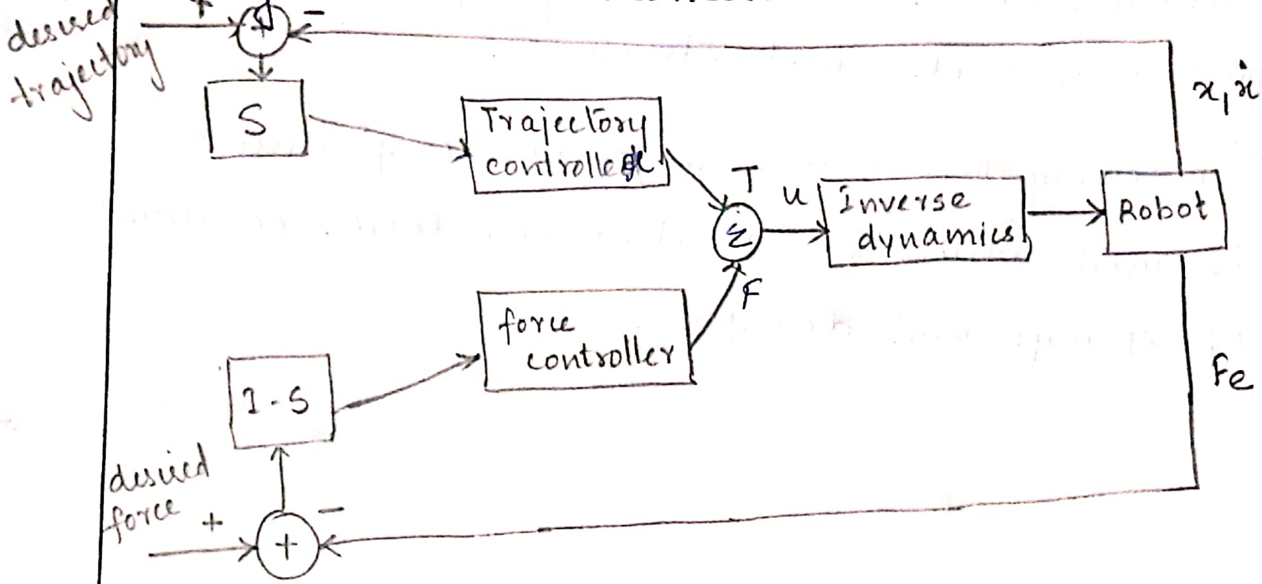
i.e. By choosing the input force.

$$F = -K_1 F_e - K_2 \dot{F}_e + \phi(t)$$

where $\phi(t)$ is an unspecified feed forward sl, tracking and disturbance rejection is achieved as in position control.

Hybrid Position / Force Control.

To implement this, control scheme, we design a position control law along force constrained directions and a force control law along position constrained directions. For each degree of freedom and implement the overall control using selection matrices.

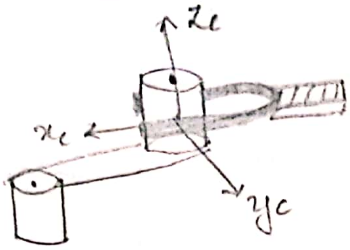


Selection matrix, S is a diagonal matrix with a 1 on the diagonal entries corresponding to degrees of freedom in the compliance frame that are to be position controlled.

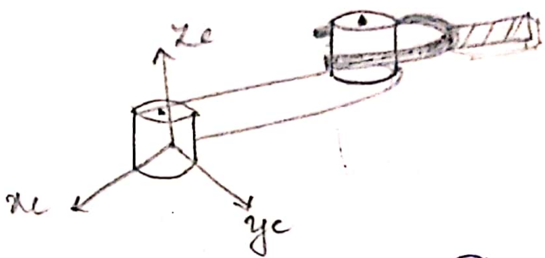
I−S is also a diagonal matrix with a 1 on the diagonal entries corresponding to degrees of freedom that are to be force controlled. In this way, each degree of freedom is uniquely specified as being either position controlled or force controlled. Any posn & force control strategy can be inserted into this control architecture.

Compliance. - Robot manipulators are very rigid by design. This rigidity is necessary for high positioning accuracy and stability. However force controlled applns are extremely difficult to accomplish with such rigid structures.

For solving this issue, a passive compliance can be used, which is a mechanical device composed of springs and dampers.



compliance frame @ handle of crank.

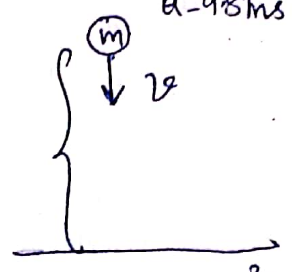


compliance frame @ axis of rotation of crank.

To describe force control tasks, it is customary to introduce a compliance frame $O_c x_c y_c z_c$ (constraint frame) in which the task to be performed is easily described.

Lagrangian Mechanics

$a = 9.8 ms^{-2}$



$L = KE - PE = T - V$

Eqn of motion $= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\dot{x}$ - velocity in generalized coordinates

x - position

$KE = \frac{1}{2} m v^2$

$PE = mgy$

$v = \dot{x}$, $y = x$

$KE = \frac{1}{2} m \dot{x}^2$

$PE = mgx$

$L = \frac{1}{2} m \dot{x}^2 - mgx$

$\frac{\partial L}{\partial x} = 0 - mg$

$\frac{\partial L}{\partial \dot{x}} = \frac{1}{2}[m\dot{x}^2] = [m \cdot 2\dot{x} + \dot{x}^2 \cdot 0] = m\dot{x}$

$\frac{dx^2}{dx} = 2x \times 1$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}(m\dot{x}) = m\ddot{x}$

$EOM = m\ddot{x} - (-mg) = 0$

$m\ddot{x} + mg = 0$

$m(g + \ddot{x}) = 0$

$\ddot{x} = -g$

$g = +9.8 ms^{-2}$

complex stms & multi-link robots.

Dynamic eqn for a 2 DOF Robots

* useful for T/F computation.
* provides relation b/w joint actuator Torques / motions of links for simulation & design of control algorithm.

Laws - Lagrangian Mechanics / Newtonian Mechanics.

Approaches Lagrange - Euler (LE) - energy based.
Newton - Euler (NE) - force-balance based

LE - systematic, real physical terms.
EOM - analytical & compact.

Lagrangian mechanics is based on 2 generalized equations. ① Linear Motion.
② Rotational Motion.

differentiation of energy terms w.r.t system's variable & time

$$F_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} \rightarrow ①$$

Sum of all external forces for a linear motion F_i

$$T_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} \rightarrow ②$$

" Torques in a rotational motion

Procedure : Derive energy equations for the s/m & differentiate the Lagrangian acc! to ① & ②

Two link Planar Manipulator

$$L = K - P \rightarrow ③ \quad K = K_1 + K_2 \rightarrow ④$$

$$K_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 \rightarrow ⑤$$

joint variables θ_1, θ_2

link lengths l_1, l_2

mass of links m_1, m_2

distance b/w joint - centre of mass r_1, r_2

linear & angular $V_1, V_2, \dot{\theta}_1, \dot{\theta}_2$

P.E = 0.

K_2 : To find

1. posn eqn for m_2 & diff: it for the velocity of m_2

$$y_2 = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$
$$= l_1 s_1 + l_2 s_{12}$$

$$x_2 = -(l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2))$$
$$= -l_1 c_1 - l_2 c_{12}.$$

$$\dot{y}_2 = l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2)$$

} → ⑥

$$V_2 = \dot{x}^2 + \dot{y}^2$$

$$v_2^2 = l_1^2\dot\theta_1^2 + l_2^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2) + 2l_1l_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2)(c_1c_{12} + s_1s_{12})$$

$$= l_1^2\dot\theta_1^2 + l_2^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2) + 2l_1l_2c_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2) \rightarrow (7)$$

$$c_1c_{12} + s_1s_{12} = c_1[c_1c_2 - s_1s_2] + s_1[s_1c_2 + c_1s_2]$$
$$= c_1^2c_2 - c_1s_1s_2 + s_1^2c_2 + s_1s_2c_1$$
$$= c_2(s_1^2 + c_1^2) = c_2$$

K.E for second mass - $[m_2]$

$$K_2 = \frac{1}{2}m_2l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2) + m_2l_1l_2c_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2) \rightarrow (8)$$

Total KE

$$K = K_1 + K_2 = \frac{1}{2}(m_1 + m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2) + m_2l_1l_2c_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2) \rightarrow (9)$$

$\sin\theta_1 = \frac{h}{l_1}$ $\quad \sin(\theta_1+\theta_2) = \frac{h}{l_2}$

PE. $\quad P.E = -mgh$

$$P_1 = -m_1gl_1\cos\theta_1 = -m_1gl_1c_1 \rightarrow (10)$$

$- m_2gh_1 - m_2gh$

$$P_2 = -m_2gl_1c_1 - m_2gl_2c_{12} \rightarrow (11)$$

$$P = P_1 + P_2 = -(m_1 + m_2)gl_1c_1 - m_2gl_2c_{12} \rightarrow (12)$$

$$L = K - P$$
$$= \frac{1}{2}(m_1 + m_2)l_1^2\dot\theta_1^2 + \frac{1}{2}m_2l_2^2(\dot\theta_1^2 + \dot\theta_2^2 + 2\dot\theta_1\dot\theta_2) + m_2l_1l_2c_2(\dot\theta_1^2 + \dot\theta_1\dot\theta_2) + (m_1 + m_2)gl_1c_1 + m_2gl_2c_{12} \rightarrow (13)$$

$$\frac{\partial L}{\partial\dot\theta_1} = (m_1 + m_2)l_1^2\dot\theta_1 + m_2l_2^2(\dot\theta_1 + \dot\theta_2) + 2m_2l_1l_2c_2\dot\theta_1 + m_2l_1l_2c_2\dot\theta_2 \rightarrow (14)$$

$$K = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\,\dot{\theta}_1\,\dot{\theta}_2)$$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - M$

$$+m_2 l_1\, l_2 C_2(\dot{\theta}_1^2 + \dot{\theta}_1\,\dot{\theta}_2) \qquad (4.9)$$

The potential energy of the system can be written as

$$P_1 = -m_1 g l_1 \,Cos\, \theta 1 \;== -m_1 g l_1 C_1 \qquad (4.10)$$

$$P_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12} \qquad (4.11)$$

$$P = P_1 + P_2 = -(m_1 + m_2)\, g l_1 C_1 - m_2 g l_2 C_{12} \qquad (4.13)$$

The Lagrangian for the system is

$$L = K - P$$

$$= \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^{\,2} + \frac{1}{2}m_2 l_2^2(\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\,\dot{\theta}_1\,\dot{\theta}_2)$$

$$+ m_2 l_1\, l_2 C_2(\dot{\theta}_1^2 + \dot{\theta}_1\,\dot{\theta}_2) + (m_1 + m_2)\, g l_1 C_1 + m_2 g l_2 C_{12}$$

$$(4.14)$$

The derivatives of the Lagrangian are,

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1+m_2)l_1^2\,\dot{\theta}_1 + m_2 l_2^2\,(\dot{\theta}_1 + \dot{\theta}_2)$$

$$+2\, m_2 l_1\, l_2 C_2\, \dot{\theta}_1 + m_2 l_1\, l_2 C_2\, \dot{\theta}_2 \qquad (4.15)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = [(m_1+m_2)l_1^2 + m_2 l_2^2$$

$$+2\, m_2 l_1\, l_2 C_2\,]\,\ddot{\theta}_1 + [m_2 l_2^2 + m_2 l_1\, l_2 C_2\,]\ddot{\theta}_2$$

$$-2\, m_2 l_1\, l_2 S_2\, \dot{\theta}_1\, \dot{\theta}_2 - m_2 l_1\, l_2 S_2\, \dot{\theta}_2^2 \qquad (4.16)$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2)\, g l_1 S_1 - m_2 g l_2 S_{12} \tag{4.17}$$

From Equation (4.2), the first equation of motion is

$$T_1 = (\{(m_1 + m_2))l_1^2 + m_2 l_2^2$$

$$+ 2m_2 l_1 l_2 C_2\}\, \ddot{\theta}_1 + \{ m_2 l_2^2 + m_2 l_1 l_2 C_2\} \ddot{\theta}_2$$

$$- 2\, m_2 l_1 l_2 S_2 \dot{\theta}_1 \dot{\theta}_2 - m_2 l_1 l_2 S_2 \dot{\theta}_2^2 + (m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12}$$

(4.18)

Similarly, $\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2\, (\dot{\theta}_1 + \dot{\theta}_2) + m_2 l_1\, l_2 C_2\, \dot{\theta}_1$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1\, l_2 C_2 \ddot{\theta}_1 - m_2 l_1\, l_2 S_2\, \dot{\theta}_1\, \dot{\theta}_2$$

$$\frac{\partial L}{\partial \theta_2} = -\, m_2 l_1\, l_2 S_2\, (\dot{\theta}_1^2 + \dot{\theta}_1\, \dot{\theta}_2) - m_2 g l_2 S_{12}$$

$$T_2 = (m_2 l_2^2 + m_2 l_1 l_2 C_2)\ddot{\theta}_1 + m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 S_2 \dot{\theta}_1^2 + m_2 g l_2 S_{12} \tag{4.19}$$

Writing these two equations in a matrix form,

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} (m_1 + m_2) l_1^2 + m_2 l_2^2 + 2m_2 l_1 l_2 C_2 & \{ m_2 l_2^2 + m_2 l_1 l_2 C_2 \} \\ (m_2 l_2^2 + m_2 l_1 l_2 C_2) & m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -m_2 l_1 l_2 S_2 \\ m_2 l_1 l_2 S_2 & 0 \end{bmatrix} \begin{matrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{matrix} + \begin{bmatrix} -m_2 l_1 l_2 S_2 & -m_2 l_1 l_2 S_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \\ \dot{\theta}_2 & \dot{\theta}_1 \end{bmatrix}$$

$$+ \begin{bmatrix} (m_1 + m_2) g l_1 S_1 + m_2 g l_2 S_{12} \\ m_2 g l_2 S_{12} \end{bmatrix} \tag{4.20}$$

Which is of the form,

$$T = A(Q)\ddot{Q} + B(Q)\dot{Q}) + C(Q) \tag{4.21}$$

where, $A(Q)$ is coupled inertia matrix, $B(Q,\dot{Q})$ is the matrix of coriolis and centrifugal forces and $C(Q)$ is the gravity matrix. T is the Input torques applied at various joints.

4.5 CONCLUDING REMARKS

The mathematical model for two DOF manipulator with rotary joints has been developed. The mathematical equations, often referred to as manipulator dynamics, are a set of equations of motion (EOM) that describe the dynamic response of the manipulator to input actuator torques. Based on the assumptions made in this section, the approximated mathematical model has been derived and it has been used for simulation and validation purposes in the subsequent chapters. Moreover, this model has been considered in different forms to check the efficiency of the proposed mathematical tool.

Robot Actuation & control using PID

PID : (Proportional plus Integral plus Derivative controller)

produces an o/p signal consisting of 3 terms

1. $\propto e(t)$ → stabilizes gain but produces a ess.

2. $\propto \int e(t)\,dt$ → reduces / eliminates ess

3. $\propto \frac{d}{dt} e(t)$ → reduces rate of change of error. overshoot ↓

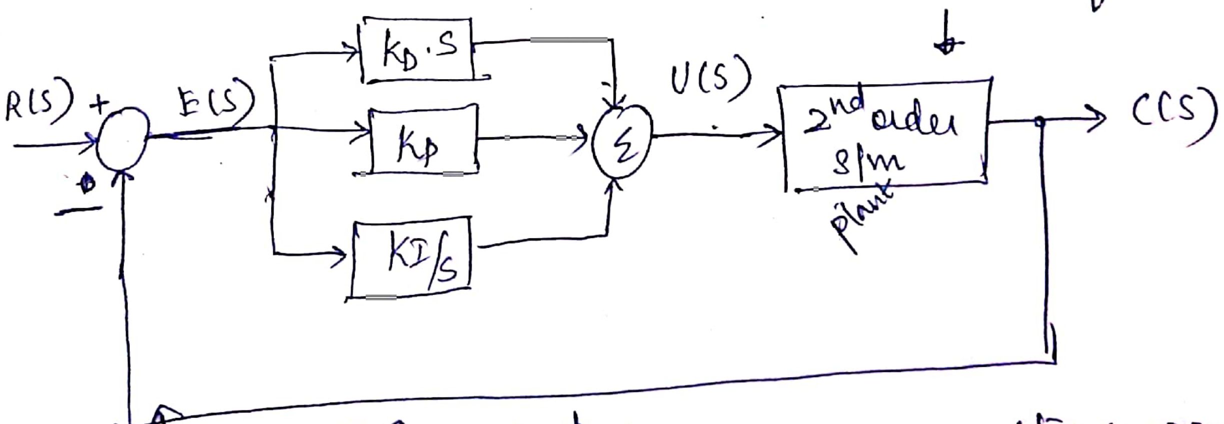
Advantage : higher stability, no offset, reduced overshoot.

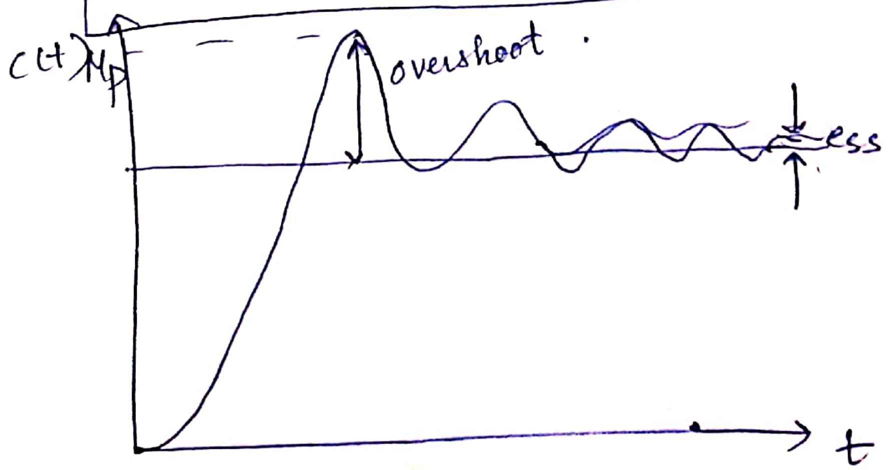
$$u(t) = K_P e(t) + K_I \int_0^t e(t)\,dt + K_D \frac{de(t)}{dt}$$

$$U(s) = K_P E(s) + K_I \frac{E(s)}{s} + K_D s \cdot E(s)$$

$$U(s) = \left[K_P + \frac{K_I}{s} + K_D \cdot s\right] E(s)$$

PID cntrl of a s/m





Step response of a 2nd order s/m