IMAGE RESTORATION

Outline

- A model of the image degradation / restoration process
- Noise models
- Restoration in the presence of noise only – spatial filtering
- Inverse filtering & Wiener filtering
- Constrained Least square filtering
- Geometric mean filter
- Geometric and spatial transformation

What is Image Restoration?

- Image restoration is to restore a degraded image back to the original image
- Image enhancement is to manipulate the image so that it is suitable for a specific application.

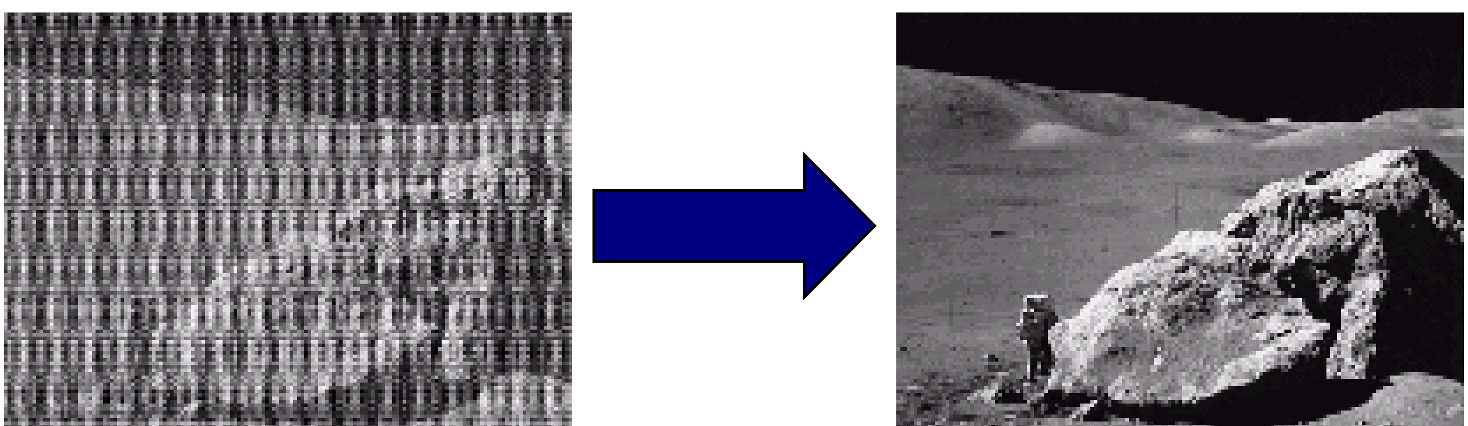
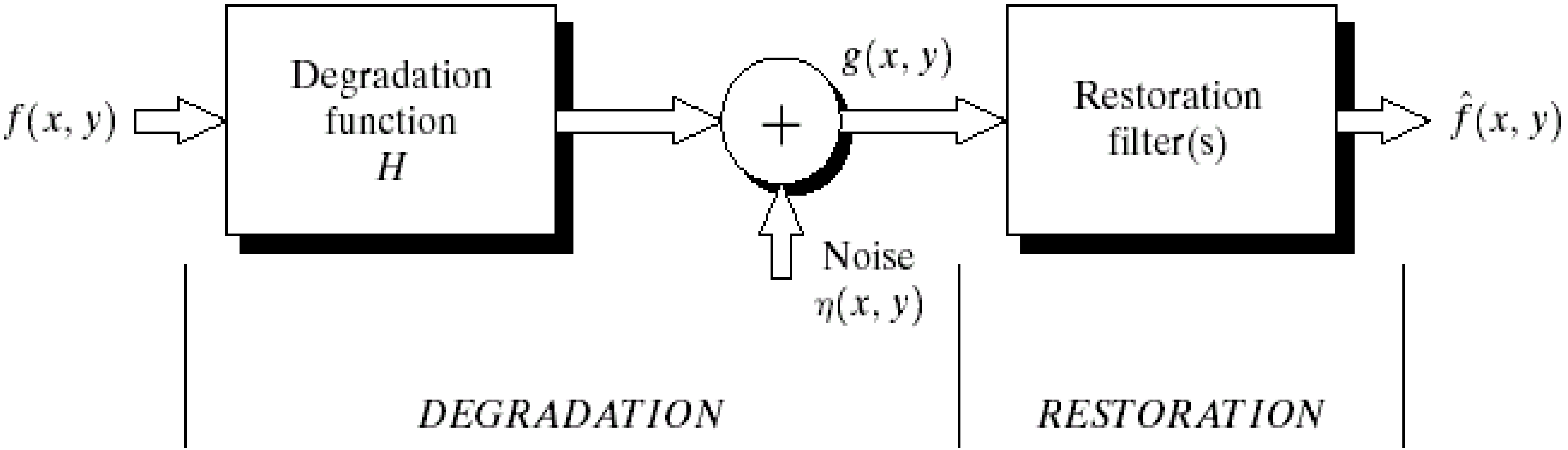


Image Restoration

Image restoration attempts to restore images that have been degraded

- Identify the degradation process and attempt to reverse it
- Similar to image enhancement, but more objective

A model of the image degradation/restoration process



$g(x,y)=f(x,y)*h(x,y)+\eta(x,y)$ – Spatial domain

$G(u,v)=F(u,v)H(u,v)+N(u,v)$ – Frequency doma

A model of the image degradation/ restoration process

- Where,
  - $f(x,y)$ - input image
  - $\hat{f}(x,y)$ - estimated original image
  - $g(x,y)$ - degraded image
  - $h(x,y)$ - degradation function
  - $\eta(x,y)$ - additive noise term

Noise models

- Sources of noise
  - Image acquisition (digitization) - Imaging sensors can be affected by ambient conditions
  - Image transmission - Interference can be added to an image during transmission
- Spatial properties of noise
  - Statistical behavior of the gray-level values of pixels
  - Noise parameters, correlation with the image
- Frequency properties of noise
  - Fourier spectrum
  - Ex. white noise (a constant Fourier spectrum)

Gaussian noise

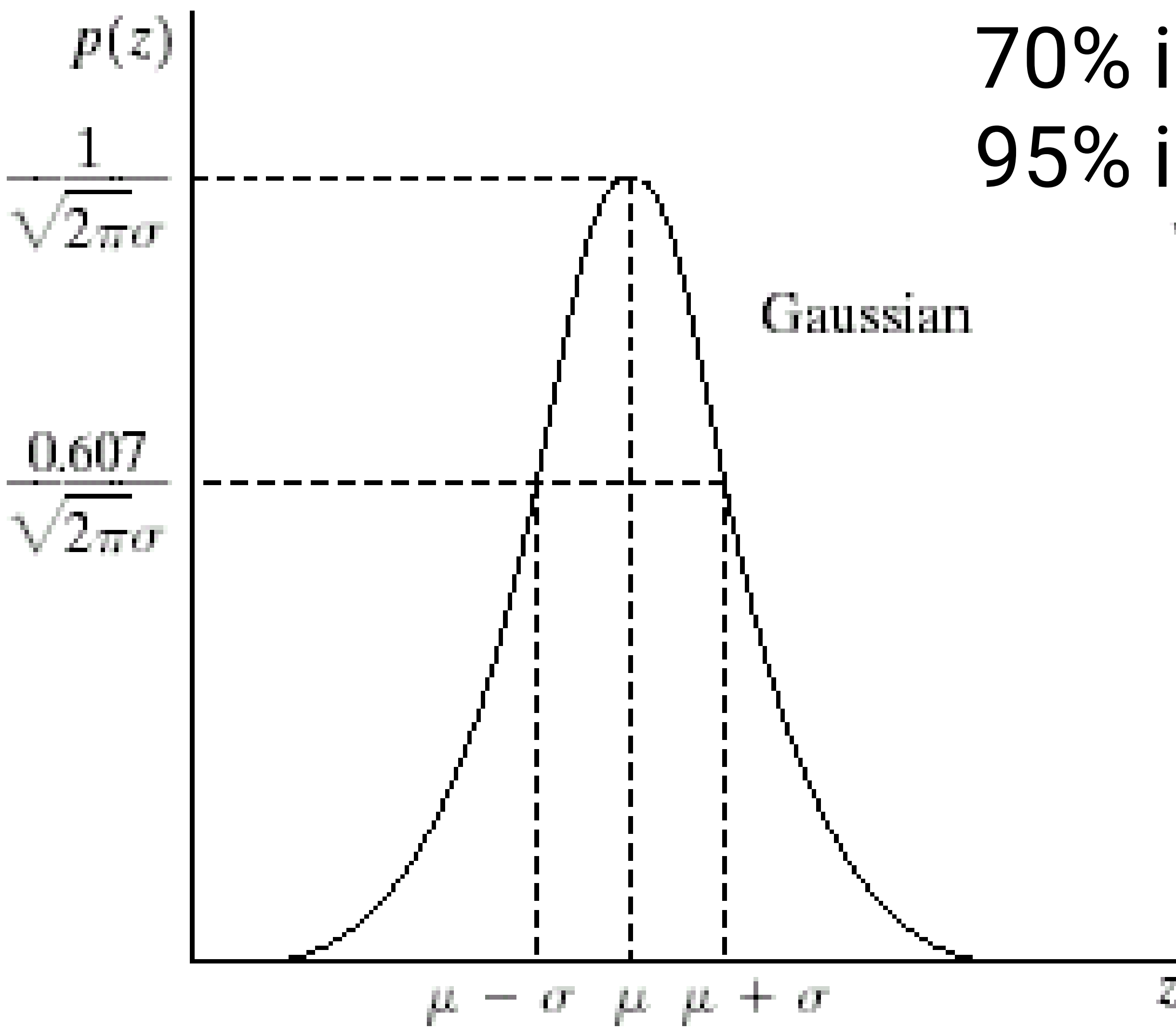
- Mathematical tractability in spatial and frequency domains
- Used frequently in practice
- Electronic circuit noise and sensor noise

$$p(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(z-\mu)^2 / 2\sigma^2}$$

Intensity mean variance

Note: $\int_{-\infty}^{\infty} p(z)\, dz = 1$

Gaussian noise PDF



70% in [(μ–σ), (μ+σ)]
95% in [(μ–2σ), (μ+2σ)]

Rayleigh noise

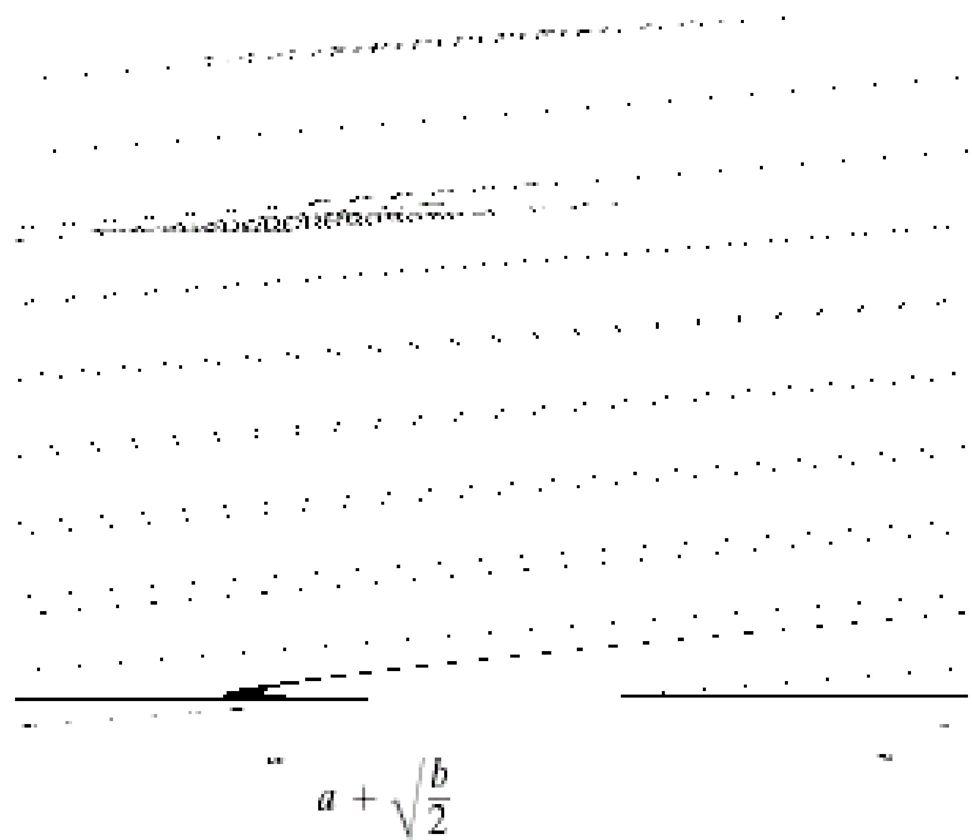
$$p(z) = \begin{cases} \frac{2}{b}(z-a)e^{-(z-a)^2/b} & \text{for } z \geq a \\ 0 & \text{for } z < a \end{cases}$$

–The mean and variance of this density are given by

$$\mu = a + \sqrt{\pi b / 4} \text{ and } \sigma^2 = \frac{b(4-\pi)}{4}$$

–a and b can be obtained through mean and variance

Rayleigh noise PDF

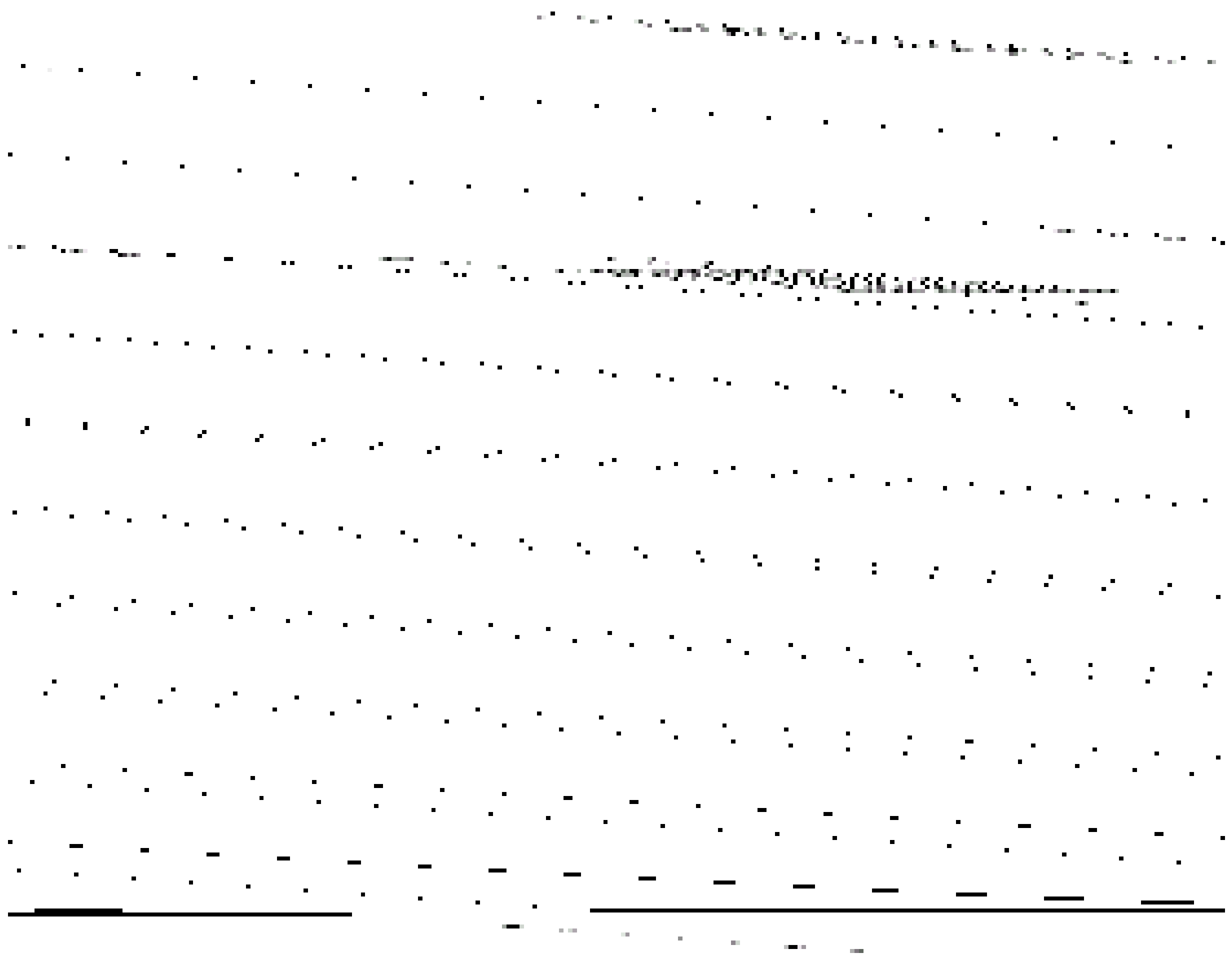


Erlang (Gamma) noise

$$p(z) = \begin{cases} \frac{a^b z^{b-1}}{(b-1)!} e^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

- The mean and variance of this density are given by $\mu = b / a$ and $\sigma^2 = \frac{b}{a^2}$
- a and b can be obtained through mean and variance

Gamma noise (PDF)



Exponential noise

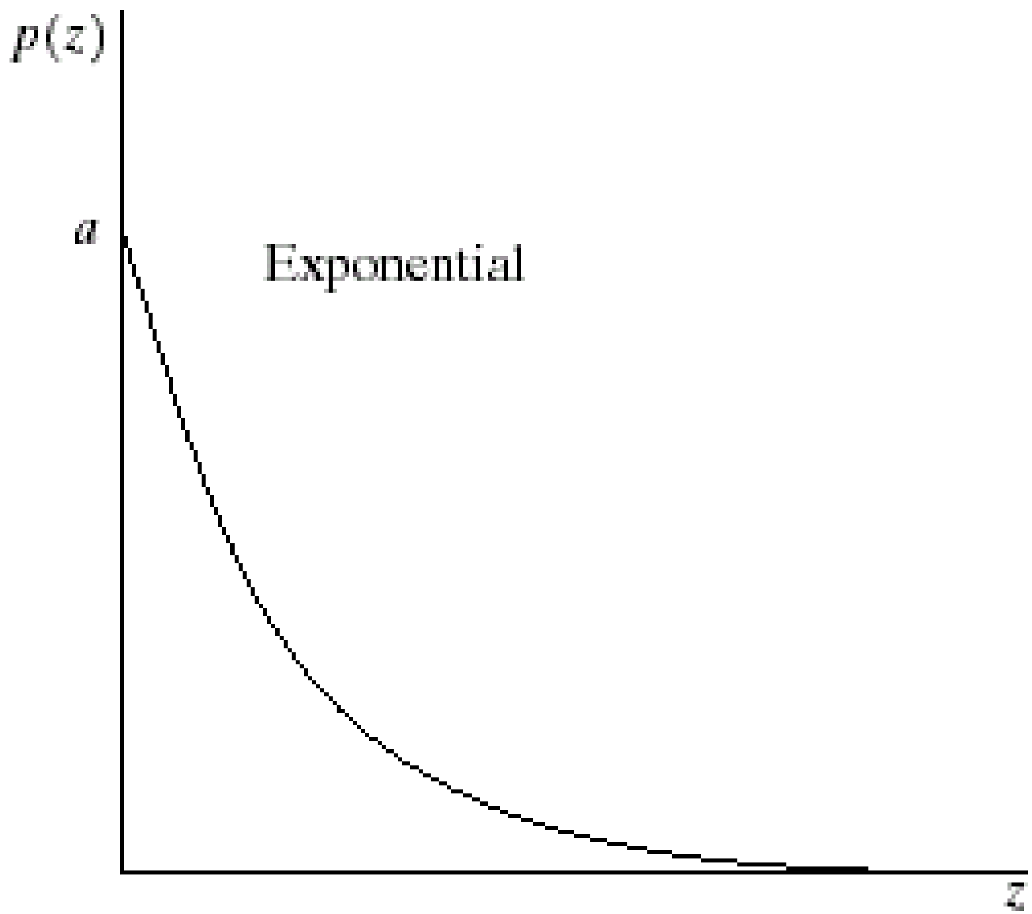
$$p(z) = \begin{cases} ae^{-az} & \text{for } z \geq 0 \\ 0 & \text{for } z < 0 \end{cases}$$

The mean and variance of this density are given by

$$\mu = 1/a \quad \text{and} \quad \sigma^2 = \frac{1}{a^2}$$

Exponential Noise PDF

Special case of Erlang PDF with b=1



Uniform noise

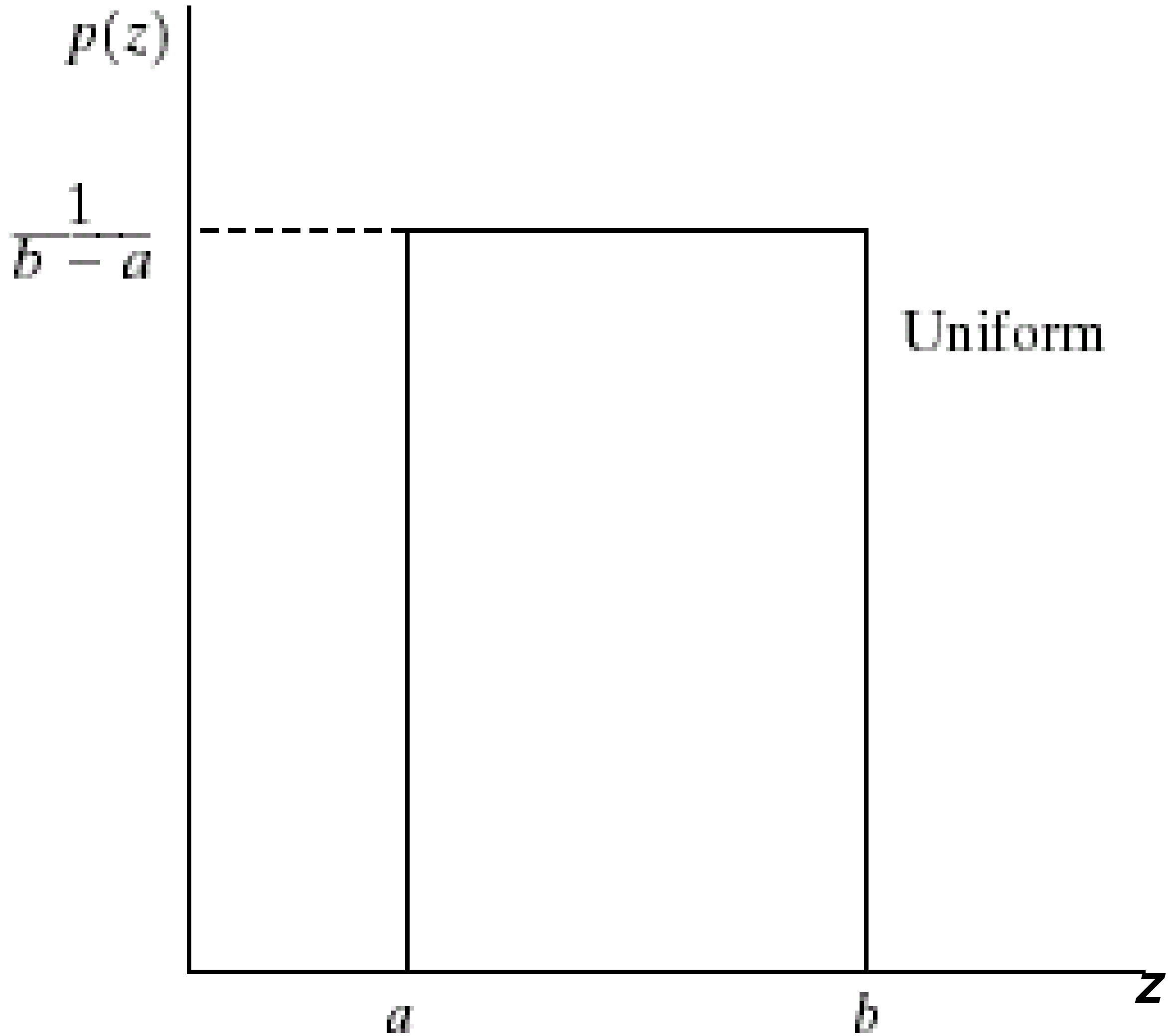
- Less practical, used for random number generator

$$p(z) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le z \le b \\ 0 & \text{otherwise} \end{cases}$$

Mean: $\mu = \frac{a+b}{2}$

Variance: $\sigma^2 = \frac{(b-a)^2}{12}$

Uniform PDF



Impulse (salt-and-pepper) nosie

- Quick transients, such as faulty switching during *imaging*

$$p(z) = \begin{cases} P_a & \text{for } z = a \\ P_b & \text{for } z = b \\ 0 & \text{otherwise} \end{cases}$$

If either P_a or P_b is zero, it is called *unipolar.*
Otherwise, it is called *bipolar.*

Impulse (salt-and-pepper) nosie PDF

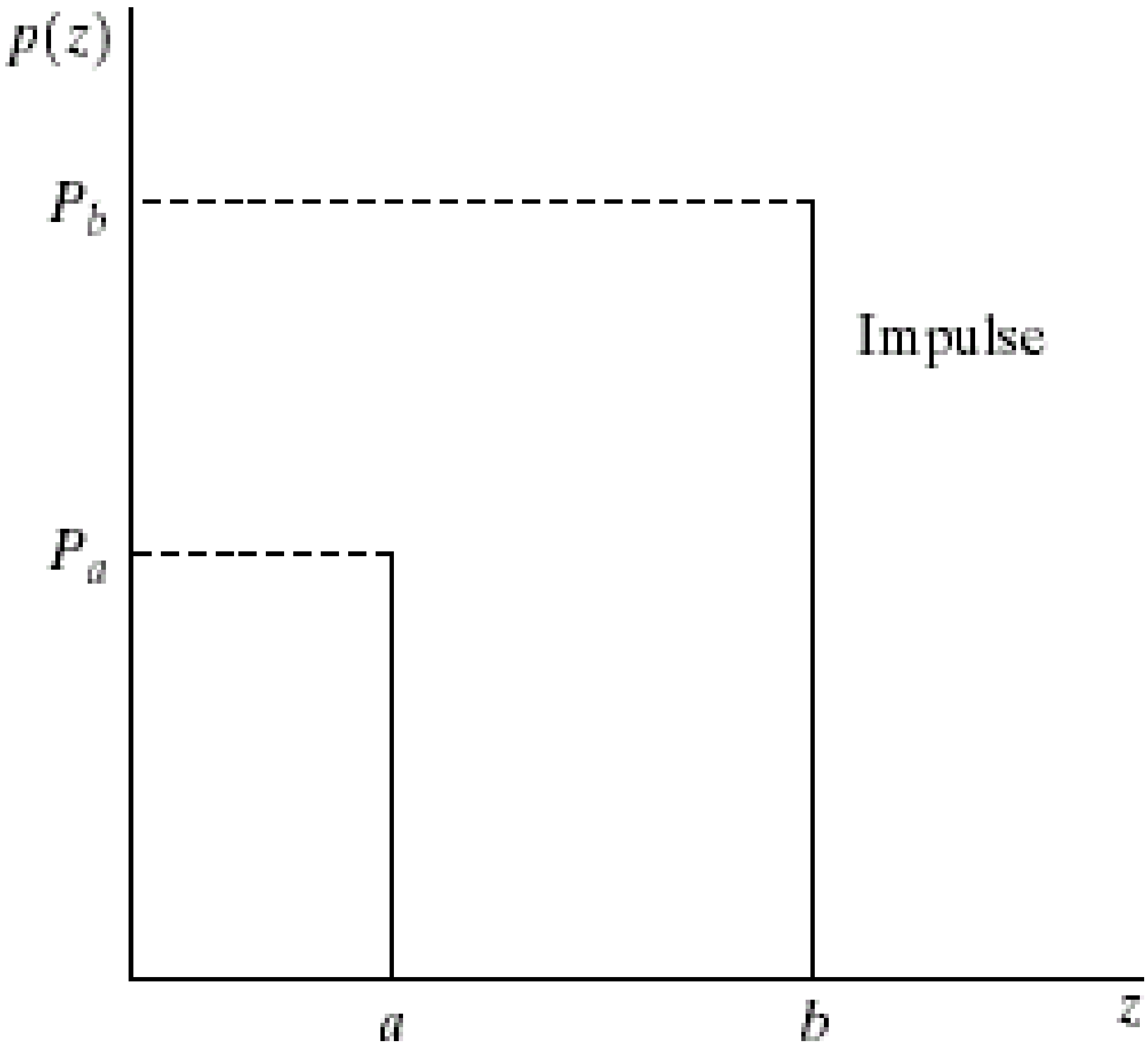
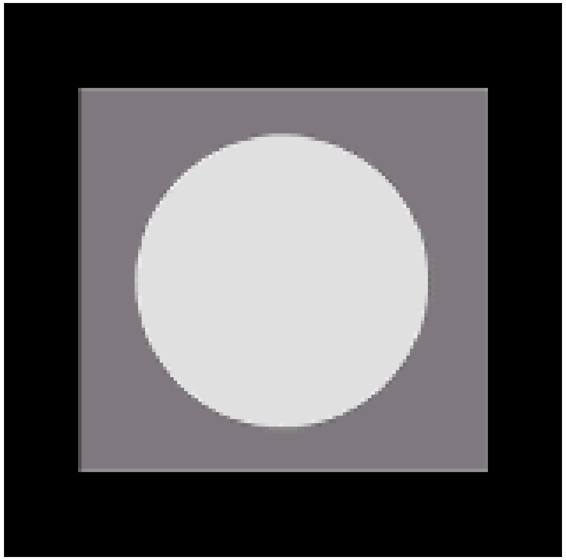


Image Degradation with Additive Noise

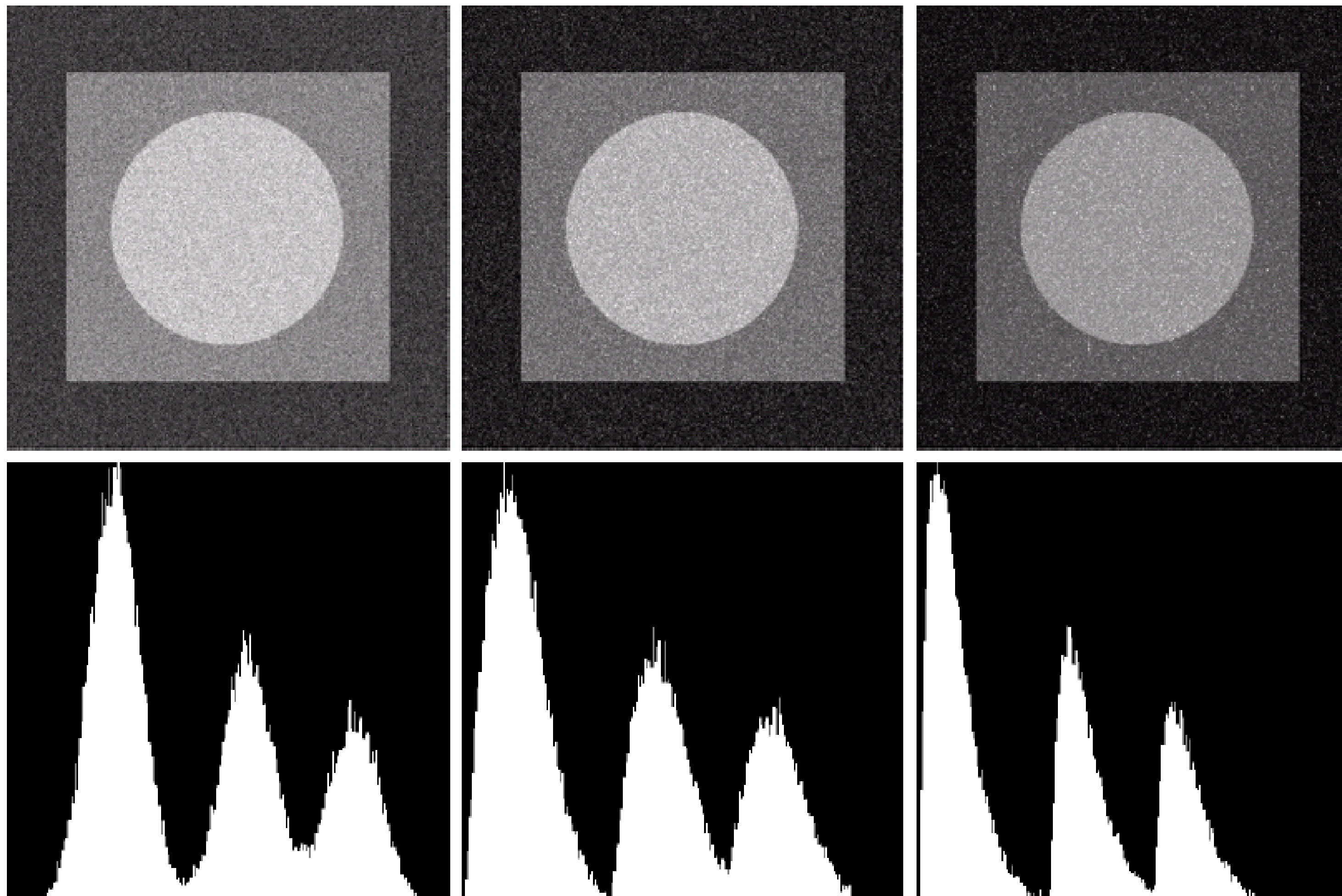


Original image

$$g(x, y) = f(x, y) + \eta(x, y)$$

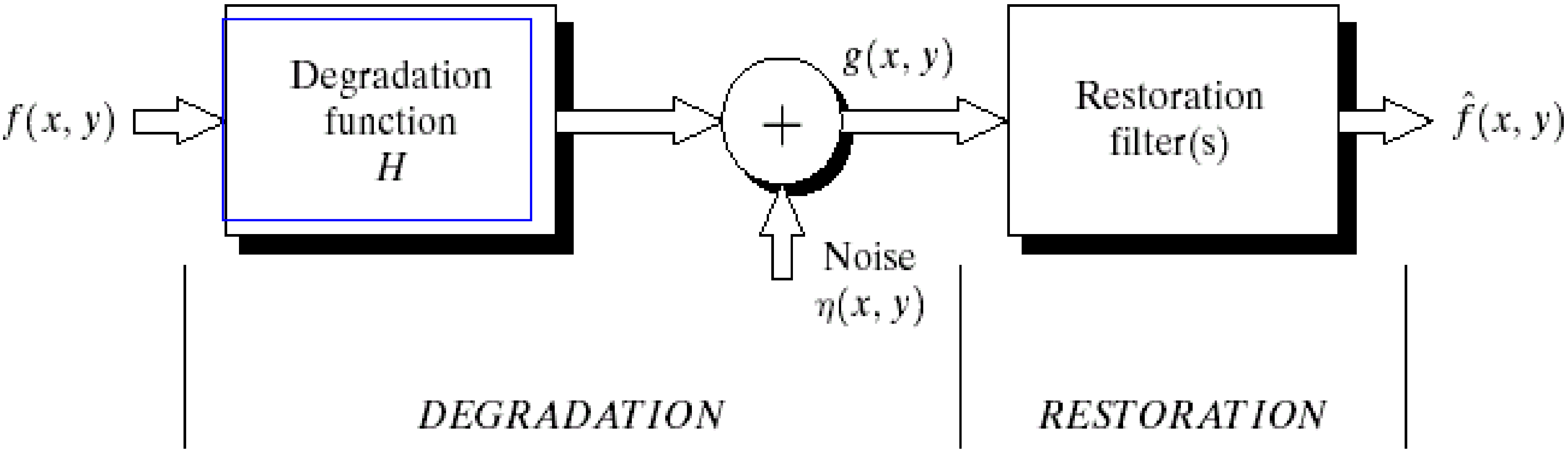
Degraded images

Histogram



Gaussian Rayleigh Gamma

A model of the image degradation / restoration process



$$\begin{cases} g(x,y)=f(x,y)*h(x,y)+\eta(x,y) \\ G(u,v)=F(u,v)H(u,v)+N(u,v) \end{cases}$$

Linear, position-invariant degradation

Properties of the degradation function H

- Linear system
  - $H[af_1(x,y)+bf_2(x,y)]=aH[f_1(x,y)]+bH[f_2(x,y)]$
- Position(space)-invariant system
  - $H[f(x,y)]=g(x,y)$ is position invariant if

$$H[f(x-\alpha, y-\beta)]=g(x-\alpha, y-\beta)$$

Estimation of Degradation Function

Degradation model:

$$g(x, y) = f(x, y) * h(x, y) + \eta(x, y)$$

or

$$G(u, v) = F(u, v) H(u, v) + N(u, v)$$

If we know exactly $h(x, y)$, regardless of noise, we can do deconvolution to get $f(x, y)$ back from $g(x, y)$.

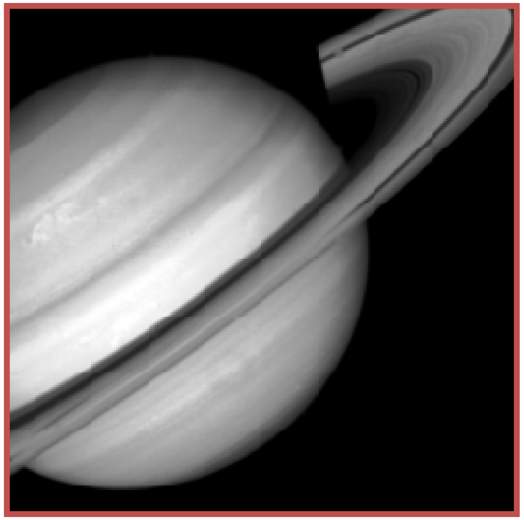
Methods:

1. Estimation by Image Observation
2. Estimation by Experiment
3. Estimation by Modeling

Estimation by Image Observation

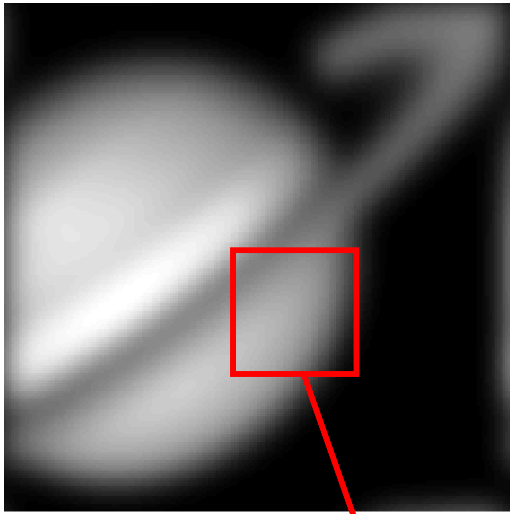
Original image (unknown)

$f(x,y)$



$f(x,y) * h(x,y)$

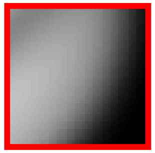
Degraded image



$g(x,y)$

Observation

Subimage



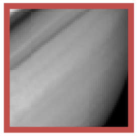
$g_s(x, y)$

DFT

$G_s(u, v)$

Restoration process by estimation

Reconstructed Subimage



$\hat{f}_s(x, y)$

DFT

$\hat{F}_s(u, v)$

Estimated Transfer function

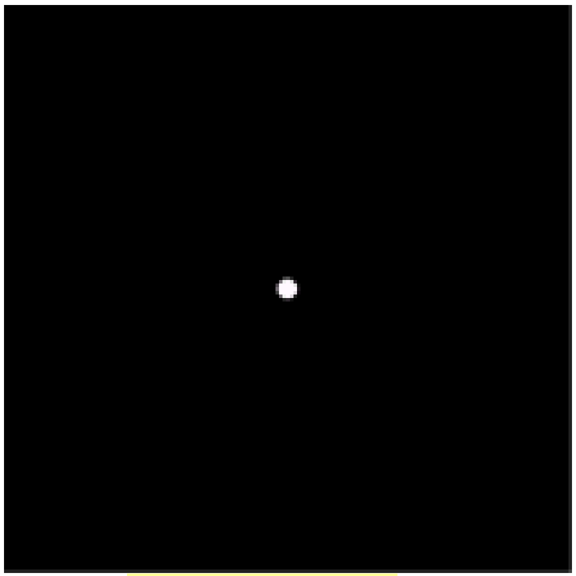
$$H(u, v) \approx H_s(u, v) = \frac{G_s(u, v)}{\hat{F}_s(u, v)}$$

This case is used when we know only $g(x,y)$ and cannot repeat the experiment!

Estimation by Experiment

Used when we have the same equipment set up

Input impulse image



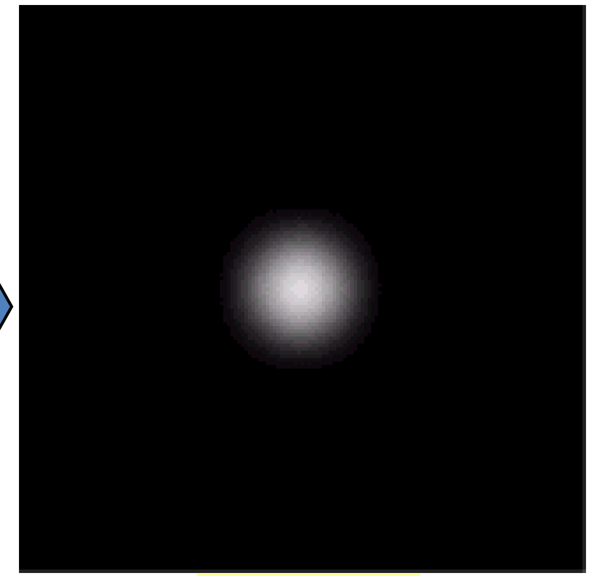
$A\,\delta(x, y)$

DFT

$DFT\ \{A\,\delta(x, y)\} = A$

System
$H(\,)$

Response image from the system



$g(x, y)$

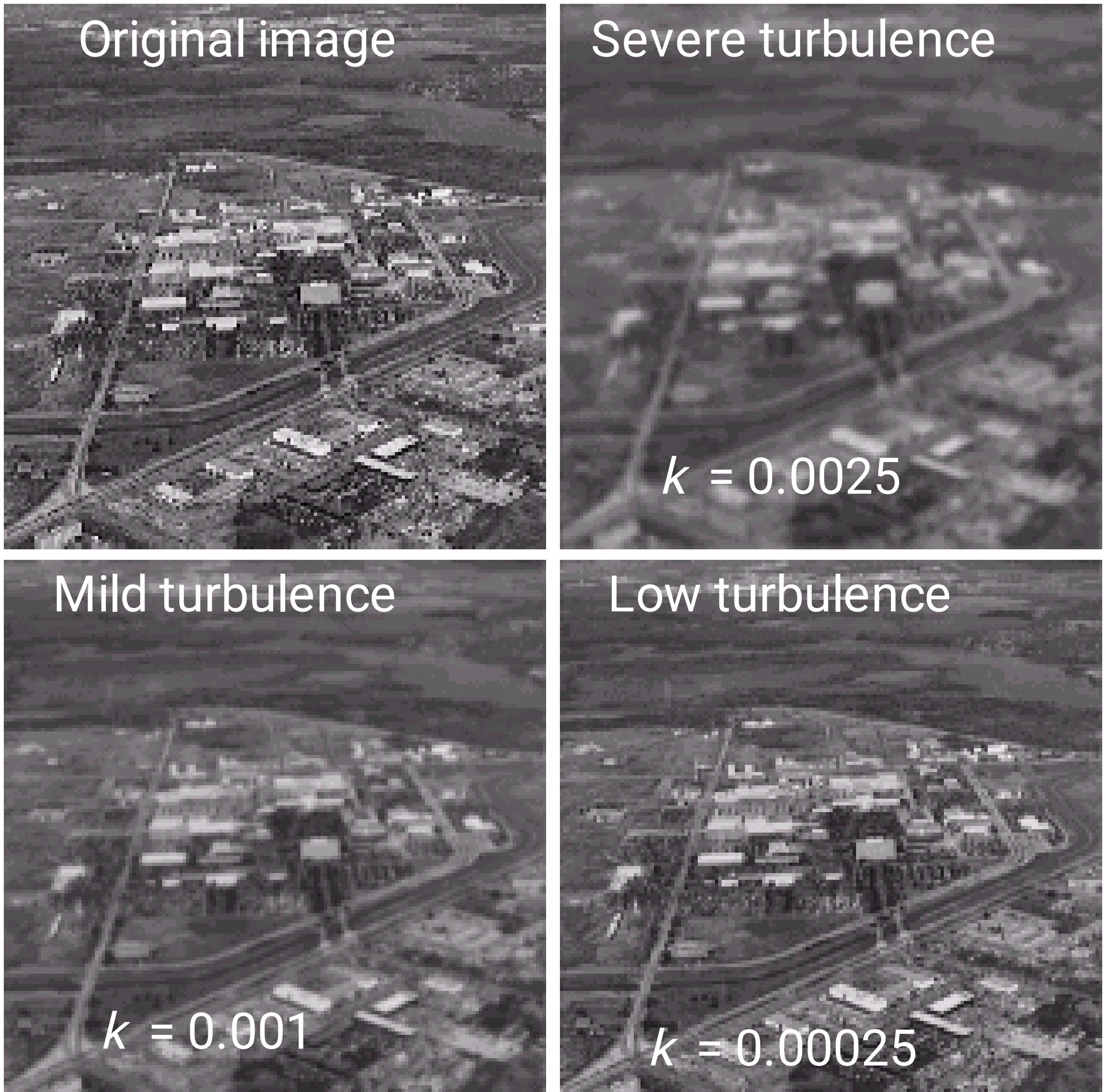
DFT

$G(u, v)$

$$H(u, v) = \frac{G(u, v)}{A}$$

Estimation by Modeling

Used when we know physical mechanism underlying the image formation process that can be expressed mathematically.



Example:

Atmospheric Turbulence model

$$H(u, v) = e^{-k(u^2+v^2)^{5/6}}$$

Estimation by Modeling: Motion Blurring

Assume that camera velocity is $(x_0(t),\ y_0(t))$

The blurred image is obtained by

$$g(x, y) = \int_0^T f(x + x_0(t),\ y + y_0(t))\, dt$$

where T = exposure time.

$$G(u, v) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)\, e^{-j2\pi(ux+vy)}\, dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_0^T f(x + x_0(t),\ y + y_0(t))\, dt\right] e^{-j2\pi(ux+vy)}\, dxdy$$

$$= \int_0^T \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x + x_0(t),\ y + y_0(t))\, e^{-j2\pi(ux+vy)}\, dxdy\right] dt$$

Estimation by Modeling: Motion Blurring (cont.)

$$G(u,v) = \int_0^T \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x + x_0(t),\ y + y_0(t))\, e^{-j2\pi(ux+vy)}\, dxdy \right] dt$$

$$= \int_0^T [F(u,v)\, e^{-j2\pi(ux_0(t)+vy_0(t))}]dt$$

$$= F(u,v) \int_0^T e^{-j2\pi(ux_0(t)+vy_0(t))}\, dt$$

Then we get, the motion blurring transfer function:

$$H(u,v) = \int_0^T e^{-j2\pi(ux_0(t)+vy_0(t))}\, dt$$

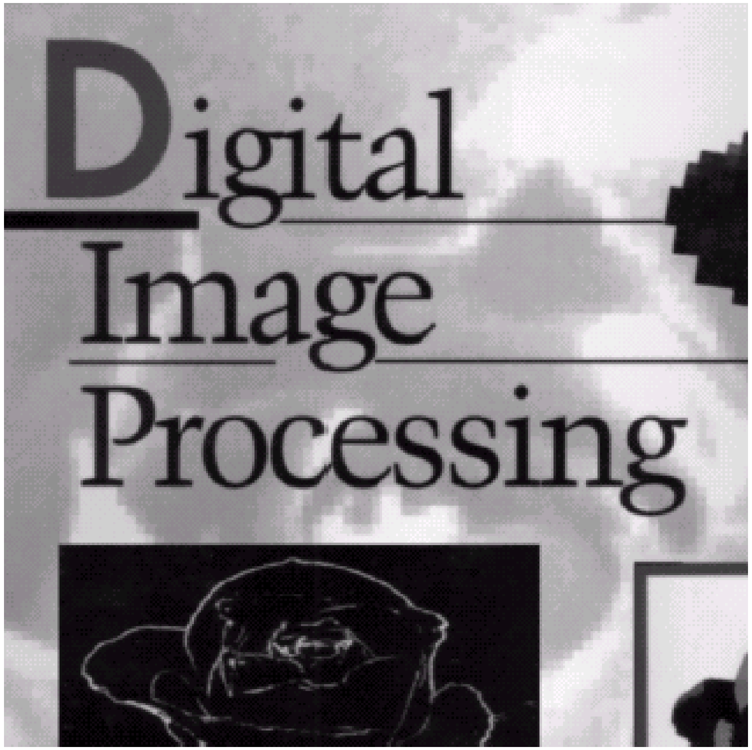
For constant motion $(x_0(t),\ y_0(t)) = (at, bt)$

$$H(u,v) = \int_0^T e^{-j2\pi(ua+vb)}\, dt = \frac{T}{\pi(ua+vb)} \sin(\pi(ua+vb))\, e^{-j\pi(ua+vb)}$$

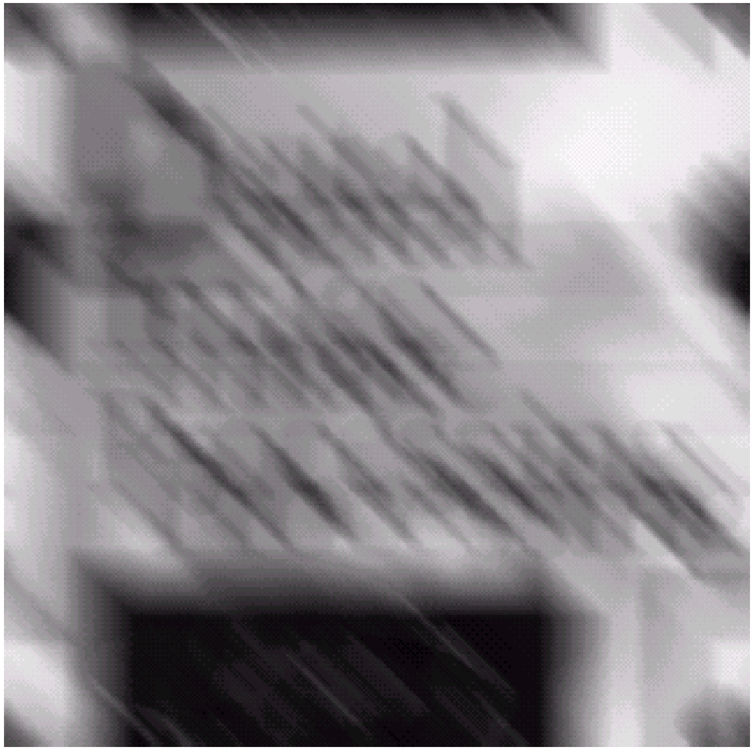
Motion Blurring Example

For constant motion

$$H(u,v) = \frac{T}{\pi(ua+vb)} \sin(\pi(ua+vb))\, e^{-j\pi(ua+vb)}$$



Original image



Motion blurred image
$a = b = 0.1, T = 1$

Inverse Filtering

From degradation model:

$$G(u,v) = F(u,v)H(u,v) + N(u,v)$$

after we obtain $H(u,v)$, we can estimate $F(u,v)$ by the inverse filter:

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)} = F(u,v) + \frac{N(u,v)}{H(u,v)}$$

Noise is enhanced when $H(u,v)$ is small.

To avoid the side effect of enhancing noise, we can apply this formulation to freq. component (u,v) with in a radius D_0 from the center of $H(u,v)$.

In practical, the inverse filter is not popularly used.

Inverse Filtering Contd...

- Divide equation one by H(u,v)
- $\frac{G(u,v)}{H(u,v)} = \frac{F(u,v)H(u,v)+N(u,v)}{H(u,v)}$ …….(2)
- We know that $\hat{F}$(u,v) = $\frac{G(u,v)}{H(u,v)}$
- Substitute $\hat{F}$(u,v) in eqn (2)
- $\hat{F}$(u,v) = F(u,v)+ $\frac{N(u,v)}{H(u,v)}$
- If noise is zero the estimated image $\hat{F}$(u,v) is equal to original image, but noise will not be properly removed in inverse filtering.

Inverse Filtering

- **Limitations:**

1. Even if the degradation function is known the undegraded image cannot be recovered exactly because N(u,v) is the random function which is not known.
2. If the degradation function has ‘0’ or small value the ratio $\frac{N(u,v)}{H(u,v)}$ easily dominates the estimate F(u,v,) one approach to get ride of 0 (or) small value problem to limits the filter frequency to the value near the origin.

WIENER FILTERING

- Inverse filtering has no explicit provision for handling noise but the wiener filtering it incorporates both degradation function, statistical characteristics of noise taken into the restoration process.
    - $e^2 = E[(f-\hat{f})^2]$
- Objective of the wiener filter is to find the estimate of uncorrupted image f, such that the mean square error is minimize the wiener filter is optimum filter
- **Diagram**

Wiener Filtering Contd...

- The error between the input signal and the estimated signal is given by the mean square error.
  - $e(x,y) = f(x,y) - \hat{f}(x,y)$
  - $E[f(x,y) - \hat{f}(x,y)^2] = 0$
- According to the principle of orthogonality the expected value of

$f(x,y) - \hat{f}(x,y)$ totally orthogonal with $g(x,y)$ is zero.

$$E[f(x,y) - \hat{f}(x,y)g(x,y)] = 0$$

$$\hat{f}(x,y) = g(x,y) * r(x,y)$$

$$E[f(x,y) - (g(x,y) * r(x,y))g(x,y)] = 0$$

$$E[f(x,y)g(x,y)] = E[(r(x,y) * g(x,y))g(x,y)]$$

$$= E\{[\sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} r(x-k, y-l)g(k,l)]g(x,y)\}$$

Wiener Filtering Contd...

- $S_{gg}(u,v) = H(u,v)H^*(u,v).\ S_{ff}(u,v)$

$$= |H(u,v)|^2\ S_{ff}(u,v)$$

$$R(u,v) = \frac{S_{fg}(u,v)}{S_{gg}(u,v)} = \frac{H^*(u,v)S_{ff}(u,v)}{|H(u,v)|^2 S_{ff}(u,v) + S\eta(u,v)}$$

With presence of noise $S_{gg}(u,v) = |H(u,v)|^2\ S_{ff}(u,v) + N(u,v)$

$$S\eta(u,v) = |N(u,v)|^2$$

$$R(u,v) = \frac{H^*(u,v)S_{ff}(u,v)}{|H(u,v)|^2 S_{ff}(u,v) + S\eta(u,v)} = \frac{\hat{F}(u,v)}{G(u,v)}$$

$$\hat{F}(u,v) = R(u,v)G(u,v)$$

Multiply and divide by $H(u,v)$ in $R(u,v)$ and sub in $\hat{F}(u,v)$

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{[H^*(u,v)H(u,v)]S_{ff}(u,v)}{[|H(u,v)|^2 S_{ff}(u,v) + S\eta(u,v)]}\right] G(u,v)$$

Wiener Filtering Contd...

- $\hat{F}(u,v) = = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + \frac{S\eta(u,v)}{S_{ff}(u,v)}}\right] G(u,v)$
- Wiener filter also know as minimum mean square filter or least mean square filter.
- Wiener filter does not have the same problem as the inverse filter unless both H(u,v) and Sη(u,v) are zero for the same value of u&v
- H(u,v) = degradation function
- H*(u,v) = complex conjugate of H(u,v)
- |H(u,v)|2 = H*(u,v) H(u,v)
- Sη(u,v) = |N(u,v)|2 = Power spectrum of the noise
- Sf(u,v) = |F(u,v)|2 = Power spectrum of an undegraded image.

Wiener Filtering

- Consideration:

1. When a noise is zero

$$\eta(x,y)=0,\ S\eta(u,v)=0$$

$$\hat{F}(u,v) = \frac{G(u,v)}{H(u,v)}$$

It reduces to inverse filtering

2. IF H(u,v)=1

$$\hat{F}(u,v)= \left[\frac{G(u,v)S_{ff}(u,v)}{S_{ff}(u,v)+S\eta(u,v)}\right]$$

$$\frac{G(u,v)\frac{S_{ff}(u,v)}{S\eta(u,v)}}{\frac{S_{ff}(u,v)}{S\eta(u,v)}+1}$$

Wiener Filtering

- Signal to Noise ratio $\frac{S_{ff}(u,v)}{S\eta(u,v)}$

3. Signal to noise ratio is greater than 1

$$\frac{S_{ff}(u,v)}{S\eta(u,v)} >> 1$$

Then $\hat{F}(u,v) =$ G(u,v) --- Here the wiener filter act as a all pass filters.

ADVANTAGES:

1. The wiener filter does not have zero value problem untill both H(u,v) and $S\eta(u,v)$ is equal to zero.
2. The result obtained by wiener filter is more closer to the original image than inverse filter.

Approximation of Wiener Filter

Wiener Filter Formula:

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + S_\eta(u,v) / S_f(u,v)} \right] G(u,v)$$

Difficult to estimate

Approximated Formula:

$$\hat{F}(u,v) = \left[\frac{1}{H(u,v)} \frac{|H(u,v)|^2}{|H(u,v)|^2 + K} \right] G(u,v)$$

In Practice, K is chosen manually to obtain the best visual result!

Constrained Least Squares Filter

Degradation model:

$$g(x, y) = f(x, y) * h(x, y) + \eta(x, y)$$

In matrix form,

$$\mathbf{g} = \mathbf{Hf} + \boldsymbol{\eta}$$

Aims to find the minimum of a criterion function

$$C = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [\nabla^2 f(x, y)]^2$$

Subject to the constraint

$$\left\| \mathbf{g} - \mathbf{H}\hat{\mathbf{f}} \right\|^2 = \left\| \boldsymbol{\eta} \right\|^2$$

where

$$\left\| \mathbf{w} \right\|^2 = \mathbf{w}^T \mathbf{w}$$

Constrained least square filter is given by,

$$\hat{F}(u, v) = \left[\frac{H^*(u, v)}{\left| H(u, v) \right|^2 + \gamma \left| P(u, v) \right|^2} \right] G(u, v)$$

where

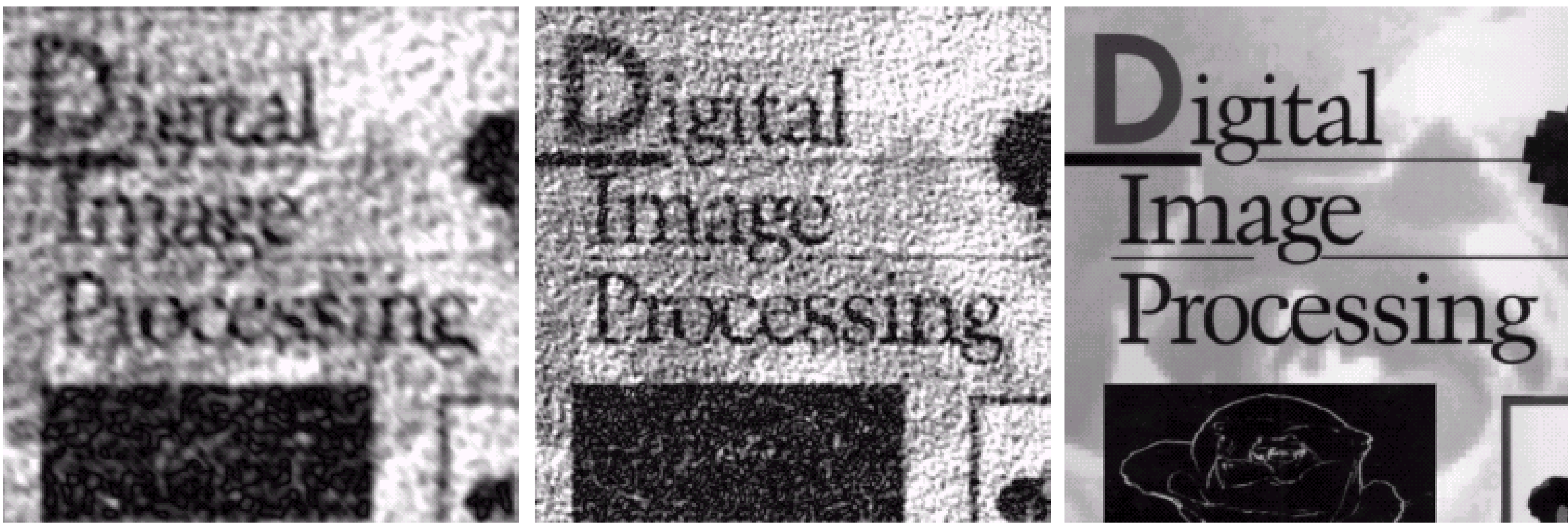
$$P(u, v) = \text{Fourier transform of } p(x, y) = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

Constrained Least Squares Filter: Example

Constrained least square filter

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v)$$

γ is adaptively adjusted to achieve the best result.



Results from the previous slide obtained from the constrained least square filter

Constrained Least Squares Filter: Adjusting γ

Define $\mathbf{r} = \mathbf{g} - \mathbf{H}\hat{\mathbf{f}}$ It can be shown that $\phi(\gamma) = \mathbf{r}^T\mathbf{r} = \|\mathbf{r}\|^2$

We want to adjust gamma so that $\|\mathbf{r}\|^2 = \|\boldsymbol{\eta}\|^2 \pm a$ → 1

where a = accuracy factor

1. Specify an initial value of γ
2. Compute $\|\mathbf{r}\|^2$
3. Stop if 1 is satisfied

Otherwise return step 2 after increasing γ if $\|\mathbf{r}\|^2 < \|\boldsymbol{\eta}\|^2 - a$

or decreasing γ if $\|\mathbf{r}\|^2 > \|\boldsymbol{\eta}\|^2 + a$

Use the new value of γ to recompute

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma|P(u,v)|^2}\right]G(u,v)$$

Constrained Least Squares Filter: Adjusting γ (cont.)

For computing $\|\mathbf{r}\|^2$:

$$\hat{F}(u,v) = \left[\frac{H^*(u,v)}{|H(u,v)|^2 + \gamma |P(u,v)|^2} \right] G(u,v)$$

$$R(u,v) = G(u,v) - H(u,v)\hat{F}(u,v)$$

$$\|\mathbf{r}\|^2 = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} r^2(x,y)$$

For computing $\|\boldsymbol{\eta}\|^2$:

$$m_\eta = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \eta(x,y)$$

$$\sigma_\eta^2 = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} [\eta(x,y) - m_\eta]^2$$

$$\|\boldsymbol{\eta}\|^2 = MN\,[\sigma_\eta^2 - m_\eta]$$

Geometric Transformation

These transformations are often called rubber-sheet transformations: Printing an image on a rubber sheet and then stretch this sheet according to some predefine set of rules.

A geometric transformation consists of 2 basic operations:

1. **A spatial transformation :**
   Define how pixels are to be rearranged in the spatially transformed image.
2. **Gray level interpolation :**
   Assign gray level values to pixels in the spatially transformed image.

Geometric Transformation : Algorithm

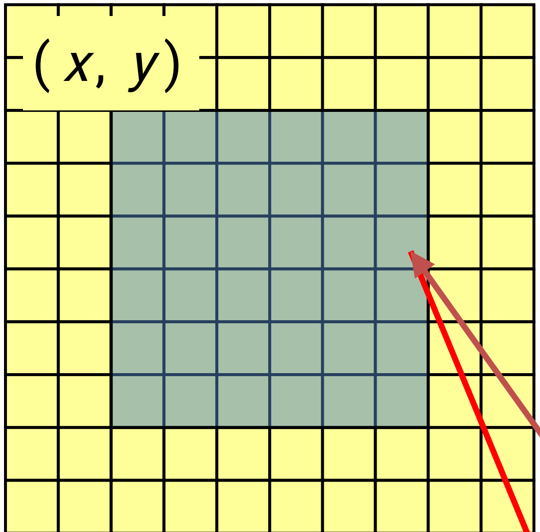
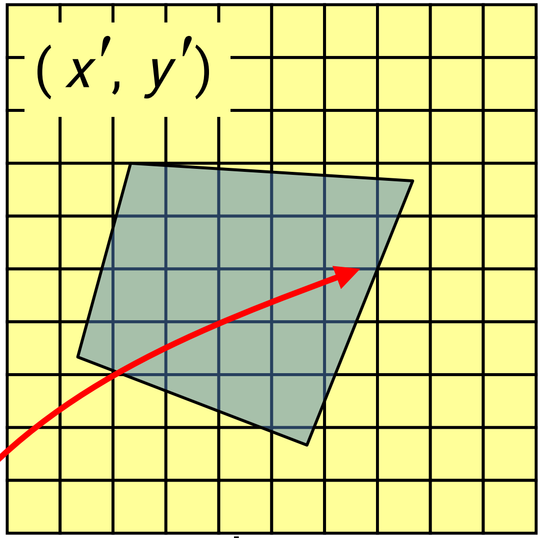


Image f to be restored



Distorted image g

1. Select coordinate (x,y) in f to be restored
2. Compute

$$x' = r(x, y)$$

$$y' = s(x, y)$$

3. Go to pixel (x', y') in a distorted image g

4. get pixel value at $g(x', y')$ By gray level interpolation

5. store that value in pixel $f(x, y)$

Spatial Transformation

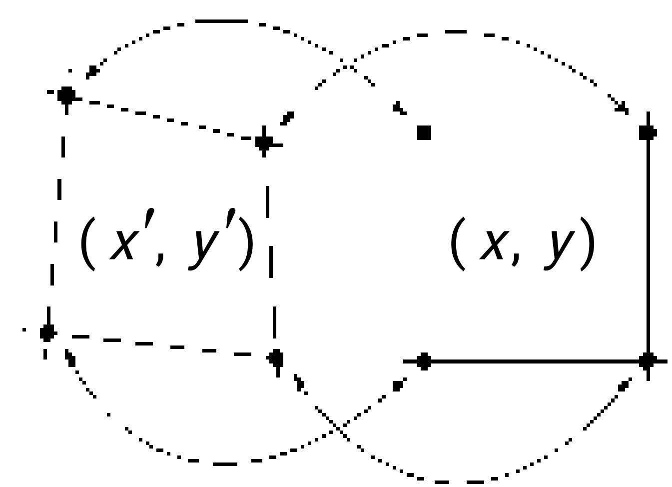
To map between pixel coordinate (x, y) of f and pixel coordinate (x', y') of g

$$x' = r(x, y) \qquad y' = s(x, y)$$

For a bilinear transformation mapping between a pair of Quadrilateral regions

$$x' = r(x, y) = c_1 x + c_2 y + c_3 xy + c_4$$

$$y' = s(x, y) = c_5 x + c_6 y + c_7 xy + c_8$$



To obtain $r(x, y)$ and $s(x, y)$, we need to know 4 pairs of coordinates (x, y) and its corresponding (x', y') which are called tiepoints.

Gray Level Interpolation: Nearest Neighbor

Since (x', y') y not be at an integer coordinate, we need to Interpolate the value of $g(x', y')$

Example interpolation methods that can be used:

1. Nearest neighbor selection
2. Bilinear interpolation
3. Bicubic interpolation