Module - V

Poynting Vector & Poynting theorem

Power transmission requires both electric & magnetic fields. The magnetic field lines are ⊥r to the electric field lines at any pt in space. Poynting vector represents the amount of energy per unit time crossing unit area at any point. The dirn of Poynting vector represents the dirn of power flow & is ⊥r to the plane containing E & H vectors.

Poynting vector (S) is given by, $\bar{P} = \bar{E} \times \bar{H}$ (W/m^2)

↳ gives the power flow through unit area at a pt.

Poynting theorem is based on law of conservation of energy in electromagnetism. It is relates the power density at a pt in an electromagnetic Poynting theorem states that the net power flowing out of the given volume 'v' is equal to the time rate of decrease in the energy stored within volume less the ~~volume~~ ohmic power dissipated.

Proof: Consider the flow of electromagnetic energy out of a surface;

$$\nabla \cdot (E \times H) = H \cdot (\nabla \times E) - E \cdot (\nabla \times H) \longrightarrow ①$$

From Maxwell's eqn;

$$\nabla \times E = -\frac{\partial B}{\partial t} = -\mu \frac{\partial H}{\partial t} \longrightarrow ②$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} = J + \epsilon \frac{\partial E}{\partial t} \longrightarrow ③$$

Substitute ② & ③ in ①;

$$\nabla\cdot(E\times H) = H\cdot\left(-\frac{\partial H}{\partial t}\right)\mu - E\cdot\left(J + \epsilon\frac{\partial E}{\partial t}\right)$$

Applying integral w.r.t dv;

$$-\int_V \nabla\cdot(E\times H)\,dv = \int_V H\cdot\left(\mu\frac{\partial H}{\partial t}\right)dv + \int_V E\cdot J\,dv + \int_V E\cdot\epsilon\frac{\partial E}{\partial t}dv$$

$$-\int_V \nabla\cdot(E\times H)\,dv = \frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right)dv + \int_V E\cdot J\,dv$$

$$\left[\begin{array}{l} H\cdot\frac{\partial H}{\partial t} = \frac{1}{2}\frac{\partial H^2}{\partial t} \\ E\cdot\frac{\partial E}{\partial t} = \frac{1}{2}\frac{\partial E^2}{\partial t}\end{array}\right]$$

$$\Rightarrow -\int_S \overset{(E\times H)}{\cancel{(\nabla\cdot P)}}\,ds = -\int_V \nabla\cdot(E\times H)\,dv \quad \text{[By Gauss divergence theorem]}$$

$$\Rightarrow \boxed{-\oint_S \bar{P}\cdot ds = \frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\mu H^2 + \frac{1}{2}\epsilon E^2\right)dv + \int_V E\cdot J\,dv}$$

$\frac{1}{2}\epsilon E^2$ – electro~~magneto~~static energy density

$\frac{1}{2}\mu H^2$ – magnetostatic energy density

First term represents the time rate of ↑ of stored electric & magnetic field energies in the volume.

Second term gives the power lost by the electromagnetic field due to motion of charges in the volume 'v' bounded by the closed surface 's'.

Hence $\oint_S P\cdot ds$ gives the total power flow out of (-ve sign) the volume through the closed surface.

Complex Poynting theorem

For sinusoidally time varying situations, since the field vectors are complex (phasors), Complex form of Poynting theorem is important.

Poynting vector, $\bar{P} = E \times \bar{H}$

Phasor forms of $\bar{E}$ & $\bar{H}$ are;

$$\bar{E} = E_0 e^{j\theta} = E_0 e^{j\omega t}$$

$$\bar{H} = H_0 e^{j\theta} = H_0 e^{j\omega t}$$

Consider real parts of phasors;

$$\Rightarrow \bar{P} = E_0 \cos(\omega t - \beta z)\, \bar{a_x} \times H_0 \cos(\omega t - \beta z)\, \bar{a_y}$$

Intrinsic impedence $\eta_0 = \frac{E_0}{H_0} \Rightarrow H_0 = \frac{E_0}{\eta_0}$

$$\Rightarrow \bar{P} = E_0 \cos(\omega t - \beta z)\, \bar{a_x} \times \frac{E_0}{\eta} \cos(\omega t - \beta z)\, \bar{a_y}$$

$$\boxed{\bar{P} = \frac{E_0^2}{\eta} \cos^2(\omega t - \beta z)\, \bar{a_z}} \quad (W/m^2)$$

Average Poynting vector / Average power density:

$$\bar{P}_{avg} = \frac{1}{T} \int_0^T \frac{E_0^2}{\eta} \cos^2(\omega t - \beta z)\, \bar{a}_z \cdot dt$$

$$= \frac{1}{T} \frac{E_0^2}{\eta} \int_0^T \frac{1 + \cos 2(\omega t - \beta z)}{2} \cdot dt$$

$$= \frac{1}{T} \frac{E_0^2}{2\eta} \left[t + \frac{\sin 2(\omega t - \beta z)}{2\omega} \right]_0^T$$

$$\overline{P}_{avg} = \frac{E_{m}^2}{2\eta T}\left[T + \frac{\sin 2(\omega T - \beta z)}{2\omega} - \frac{\sin(-2\beta z)}{2\omega}\right]$$

$\omega = 2\pi f$

$\cancel{2\pi f} \; f$

$T = \frac{2\pi}{\omega}$

$$T = \frac{2\pi}{\omega}$$

$$\overline{P}_{avg} = \frac{E_0^2}{2\eta T}\left[T + \frac{\sin(4\pi - 2\beta z) + \sin(2\beta z)}{2\omega}\right]$$

$$\Rightarrow \overline{P}_{avg} = \frac{E_0^2}{2\eta T}\left[T + \frac{1}{2\omega}\left[\sin(4\pi - 2\beta z) + \sin(2\beta z)\right]\right]$$

$$= \frac{E_0^2}{2\eta T}\, T$$

$$\Rightarrow \boxed{\overline{P}_{avg} = \frac{E_0^2}{2\eta}} \quad (W/m^2)$$

Q: In free space $E = 150 \cos(\omega t - \beta z)\,\overline{a_x}$ V/m. Calculate total power passing through a rectangular area of side 30mm & 15mm in $z=0$ plane. Assume $\eta_0 = 120\pi\ \Omega$.

given $E = 150 \cos(\omega t - \beta z)\,\overline{a_x}$

total power through unit area,

$$\overline{P}_{avg} = \frac{E_0^2}{2\eta_0} = \frac{150^2}{2 \times 120\pi} = 29.84\ W/m^2$$

Total power $= \overline{P}_{avg} \times$ area

$= 29.84 \times 30 \times 10^{-3} \times 15 \times 10^{-3}$

$= 13.428$ mW

Q. An uniform plane wave of freq 5 MHz, $S_{avg} = \frac{E_0^2}{2\eta_0}$ = 1.5 W/m². If the medium is loss less with relative permeability $\mu_r = 2$ & relative permittivity $\epsilon_r = 3$, find; i) velocity of propagation

ii) wavelength

iii) characteristic impedence

iv) rms value of electric field.

$$\bar{P}_{avg} = \frac{E_0^2}{2\eta_0} = 1.5 \ W/m^2 \quad , \quad f = 5 \ MHz$$
$$\mu_r = 2 \ , \ \epsilon_r = 3$$

i) $$V = \frac{c}{\sqrt{\epsilon_r \mu_r}} = \frac{3\times 10^8}{\sqrt{2\times 3}} = 1.225\times 10^8 \ m/s$$

ii) $$\lambda = \frac{V}{f} = \frac{1.225\times 10^8}{5\times 10^6} = 24.49 \ m$$

iii) $$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{4\pi\times 10^{-7}\times 2}{8.85\times 10^{-12}\times 3}} = 307.67 \ \Omega$$

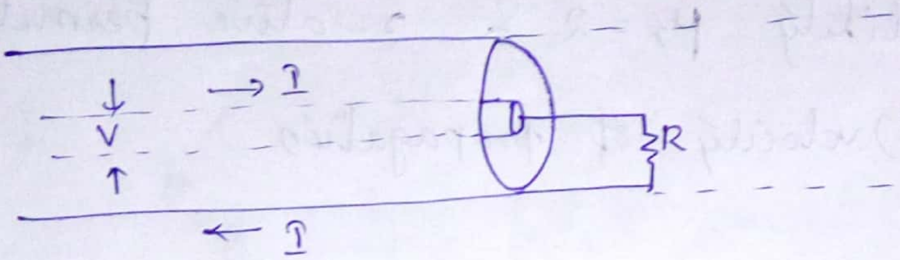
$(\mu = \mu_0 \mu_r)$

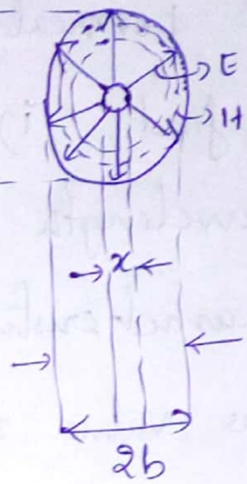
iv) $$E_{rms} = \sqrt{\bar{P}_{avg}\cdot 2\eta_0} \quad \left[E_0^2 = \bar{P}_{avg}\cdot 2\eta_0 \right]$$

$$= \sqrt{1.5\times 2\times 307.67} = 30.38 \ V/m$$

Scanned by CamScanner

Power flow in a co axial cable:





a - radius of inner conductor
b - radius of outer conductor

Consider a co axial cable where power is transferred to the load resistance along the cable.

A dc current V exists b/w the two conductors due to the current I that flows b/w the inner & outer conductors.

H will be directed in the circular path about the axis.

The current enclosed in the region b/w the two conductors is equal to the mmf given by Ampere circuital law;

$$\oint H \cdot dl = I$$

$$\bar{H} = \frac{I}{2\pi r} \longrightarrow ①$$

If q is charge per unit length, then P.D b/w inner & outer conductor of the coaxial cable is;

$$V = \frac{q}{2\pi\epsilon} \log\left(\frac{b}{a}\right) \longrightarrow ②$$

Ilrly, $\bar{E} = \dfrac{q}{2\pi\epsilon r}$ (for a co axial cable) $\rightarrow$ ③

from ② $\rightarrow$ $q = \dfrac{V \cdot 2\pi\epsilon}{\log\left(\frac{b}{a}\right)}$

$$\Rightarrow \bar{E} = \frac{V}{r\log\left(\frac{b}{a}\right)} \longrightarrow ④$$

~~Bu~~ Poynting vector $\bar{P} = \bar{E} \times \bar{H}$

Total power flow along the cable is obtained by applying Poynting theorem,

$$\text{Total power flow} = \oint_s \bar{P} \cdot ds$$

$$ds = 2\pi r\, dr$$

from ① & ④

$$\Rightarrow W = \oint_s \bar{P} \cdot ds = \int_s \frac{V}{r\log\left(\frac{b}{a}\right)} \times \frac{I}{2\pi r} \cdot 2\pi r\, dr$$

$$= \int_a^b \frac{VI}{r\log\left(\frac{b}{a}\right)}\, dr = \frac{VI}{\log\left(\frac{b}{a}\right)} \int_a^b \frac{1}{r}\, dr$$

$$= \frac{VI}{\log\left(\frac{b}{a}\right)} \left(\log r\right)_a^b$$

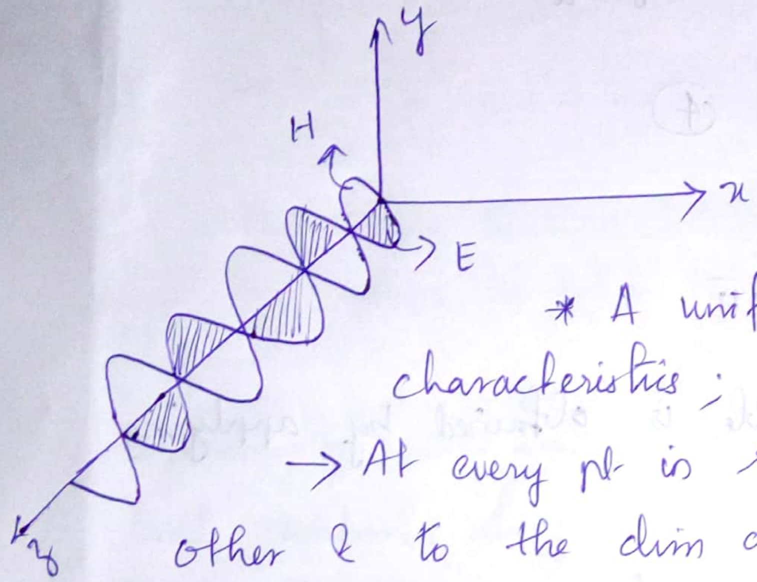
$$\Rightarrow \boxed{W = VI} \rightarrow \text{Power flow in a co axial cable.}$$

Electro magnetic waves:

Uniform Plane waves

$E = E_x a_x$

$H = H_y a_y$



* A uniform plane wave has following characteristics;

→ At every pt in space E & H are ⊥r to each other & to the dirn of travel.

→ Field vary harmonically with time & at the same freq everywhere in space.

→ Each field has same dirn, mag & phase at every pt in any plane ⊥r to the dirn of wave propagation.

i.e, if the dirn of propagation is along z-axis, then the comp of H & E are transversed to the dirn of propagation. Such waves are referred to as transverse electromagnetic waves (TEM waves).

Time varying electric & magnetic fields lead to time varying magnetic & electric in space.

From Maxwell's equations;

$$\nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \rightarrow ①$$

$$\nabla \times H = J + \frac{\partial D}{\partial t} \rightarrow ②$$

~~$\nabla \times$~~

$$\nabla \cdot E = 0 \quad (\nabla \cdot D = 0) \rightarrow ③$$

$$\nabla \cdot B = 0 \quad (\nabla \cdot H = 0) \rightarrow ④$$

Taking curl of eqn ①;

$$\nabla \times \nabla \times E = -\nabla \times \left(\mu \frac{\partial H}{\partial t}\right) = -\mu_0 \frac{\partial}{\partial t} (\nabla \times H)$$

$$\Rightarrow \nabla(\nabla \cdot E) - \nabla^2 E = -\mu \frac{\partial}{\partial t} (\nabla \times H)$$

~~Consider LHS~~

$$\nabla(\nabla \cdot E) = 0 \quad [\text{from } ③]$$

$$\cancel{\nabla^2 E = -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial D}{\partial t}\right) \text{ from } ②}$$

from ②:

$$-\mu \frac{\partial}{\partial t}\left[\sigma E + \frac{\partial}{\partial t}(\epsilon E)\right] = -\left[\mu \sigma \frac{\partial E}{\partial t} + \mu \epsilon \frac{\partial^2 E}{\partial t^2}\right]$$

$$\Rightarrow \boxed{\nabla^2 E = \mu \sigma \frac{\partial E}{\partial t} + \mu \epsilon \frac{\partial^2 E}{\partial t^2}}$$

General wave eqn

Ily, Applying curl to eqn ②;

$$\boxed{\nabla^2 H = \mu \sigma \frac{\partial H}{\partial t} + \epsilon \mu \frac{\partial^2 H}{\partial t^2}}$$