LAPLACE TRANSFORMS

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1 Laplace Transforms

1.1 Definition

Let a function $f(t)$ be continuous and defined for a positive value of t The Laplace transform of $f(t)$ associate a function s defined by $\phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

Here $\phi(s)$ is said to be the Laplace transform of $f(t)$ and it is denoted by $L(f(t))$, *or* $L(f)$ that is $L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

1. Find the Laplace transform of $ft) = \begin{cases} e^t, & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

   Ans.

$$\begin{aligned} L(f(t)) &= \int_0^\infty e^{-st}\, dt \\ &= \int_0^1 e^{-st} e^t\, dt + \int_1^\infty e^{-st} 0\, dt \\ &= \int_0^1 e^{-(s-1)t}\, dt = \left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^1 \\ &= \frac{-(s-1)}{-(s-1)} - \frac{1}{-(s-1)} = \frac{e^{(1-s)}}{1-s} - \frac{1}{1-s} \end{aligned}$$

2. Find the Laplace transform of $f(t) = \begin{cases} \cos t, & 0 < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

   Ans.

$$\begin{aligned} L(f(t)) &= \int_0^\infty e^{-st} f(t)\, dt = \int_0^{2\pi} e^{-st} \cos t\, dt \\ &= \left[\frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\right]_0^{2\pi} \\ &= \frac{e^{-2\pi s}}{s^2+1}(-s) - \frac{1}{s^2+1}(-s) \\ &= (1 - e^{-2\pi s})\frac{s}{s^2+1} \end{aligned}$$

Some basic Laplace Formulas

1. $L(1) = \frac{1}{s}$
2. $L(t) = \frac{1}{s^2}$
3. $L(t^n) = \frac{n!}{s^{n+1}}$ where n is an integer
4. $L(e^{at}) = \frac{1}{s-a}$
5. $L(\sin at) = \frac{a}{s^2+a^2}$
6. $L(\cos at) = \frac{s}{s^2+a^2}$
7. $L(\sinh at) = \frac{a}{s^-a^2}$
8. $L(\cosh at) = \frac{s}{s^2-a^2}$
9. $L[af(t) + bg(t)] = aL(f(t)) + bL(g(t))$
10. $L[af(t) - bg(t)] = aL(f(t)) - bL(g(t))$

Some basic Useful Results

1. $\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right)$
2. $\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right)$
3. $\sin^2 x = \frac{1-\cos 2x}{2}$
4. $\cos^2 x = \frac{1+cos 2x}{2}$
5. $\sin^3 x = \frac{1}{4}\left[3\sin x - \sin 3x\right]$
6. $\cos^3 x = \frac{1}{4}\left[\cos 3x - 3\cos x\right]$
7. $\sin x\cos y = \frac{1}{2}\left[\sin(x+y)+\sin(x-y)\right]$
8. $\cos x\sin y = \frac{1}{2}\left[\sin(x+y)-\sin(x-y)\right]$
9. $\cos x\cos y = \frac{1}{2}\left[\cos(x+y)+\cos(x-y)\right]$
10. $\sin x\sin y = \frac{1}{2}\left[\cos(x-y)-\cos(x+y)\right]$

1.2 Problems Based On Transforms of Elementary Function

1. Find the Laplace transforms of following functions

   (a) $\sin 3t\cos 2t$

   (b) $\cos^3 2t$

   (c) $\sin 3t\sin 2t$

   (d) $sin^2 3t$

   (e) $e^{-4t} - 6t^2 + 4\sin 2t$

   Ans.

   (a)

$$\begin{aligned} L(\sin 3t\cos 2t) &= \frac{1}{2}\{L(\sin 5t) + L(\sin t)\} \\ &= \frac{1}{2}\{\frac{5}{s^2+25} + \frac{1}{s^2+1}\} \\ &= \frac{3(s^2+5)}{(s^2+1)(s^2+25)} \end{aligned}$$

   (b)

$$\begin{aligned} L(\sin^2 3t) &= L\left[\frac{1-\cos 6t}{2}\right] = \frac{1}{2}\{L(1) - L(\cos 6t)\} \\ &= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+36}\right\} = \frac{18}{s(s^2+36)} \end{aligned}$$

   (c)

$$\begin{aligned} L(\sin 3t\sin 2t) &= \frac{1}{2}\left\{L(\cos t) - L(\cos 5t)\right\} \\ &= \frac{1}{2}\left\{\frac{s}{s^2+1} - \frac{s}{s^2+25}\right\} = \frac{24}{(s^2+1)(s^2+25)} \end{aligned}$$

   (d)

$$\begin{aligned} L(sin^2 3t) &= \frac{1}{2}\{1 - \cos 6t\} \\ &= \frac{1}{2}\left\{\frac{1}{s} - \frac{s}{s^2+36}\right\} = \frac{18}{s(s^2+36} \end{aligned}$$

(e)

$$\begin{aligned} L(e^{-4t} - 6t^2 + 4\sin 2t) &= L(e^{-4t} - 6L(t^2) + 4L(\sin 2t) \\ &= \frac{1}{s+4} - 6\frac{2!}{s^3} + 4\frac{2}{s^2+4} \\ &= \frac{1}{s+4} - \frac{12}{s^3} + \frac{8}{s^2+4} \end{aligned}$$

1.3 First Shifting Theorem

If $L(f(t)) = \phi(s)$, then $L(e^{at}f(t)) = \phi(s-a)$

1. Find the Laplace Transforms of the following

   (a) $e^{-3t}t^3$

   (b) $e^{-2t}\cos^2 t$

   (c) $\sinh at \cdot \sin at$

   (d) $e^{-2t}[\cos 4t + 3\sin 4t]$

   (e) $(t+1)^2 e^t$

   Ans.

   (a)

$$\begin{aligned} We\ have\ L(t^3) &= \frac{3!}{s^4} \\ L(e^{-3t}t^3) &= \frac{6}{(s+3)^4}[\text{ replace s by s+3}] \end{aligned}$$

   (b)

$$\begin{aligned} We\ have\ L(\cos 2t) &= \frac{s}{s^2+4} \\ L(e^{-2t}\cos^2 t) &= \frac{s+2}{(s+2)^2+4}[\text{ replace s by s+2}] \end{aligned}$$

   (c) We have $\sinh at = \frac{e^{at}-e^{-at}}{2}$ and $L(\sin at) = \frac{a}{s^2+a^2}$

$$\begin{aligned} L(\sinh at \sin at) &= L(\frac{e^{at}-e^{-at}}{2}\sin at) \\ &= \frac{1}{2}\left\{e^{at}\sin at - e^{-at}\sin at\right\} - - - - - - - (1) \\ L(e^{at}\sin at) &= \frac{a}{(s-a)^2+a^2} \qquad L(e^{-at}\sin at) = \frac{a}{(s+a)^2+a^2} \\ \text{form (1) } L(\sinh at \sin at) &= \frac{1}{2}\left\{\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right\} \end{aligned}$$

   (d)

$$\begin{aligned} L(e^{-2t}[\cos 4t + 3\sin 4t]) &= L(e^{-2t}\cos 4t) + L(3e^{-2t}\sin 4t) \\ &= \frac{s+2}{(s+2)^2+16} + 3\frac{4}{(s+2)^2+16} \\ &= \frac{s+2}{(s+2)^2+16} + \frac{12}{(s+2)^2+16} = \frac{s+14}{s^2+4s+20} \end{aligned}$$

(e)

$$\begin{aligned} L((t+1)^2e^t) &= L((t^2+2t+1)e^t) \\ &= L(e^t t^2) + 2L(e^t t) + L(e^t) \\ &= \frac{2}{(s-1)^3} + \frac{1}{(s-1)^2} + \frac{1}{s-1} = \frac{s^2+1}{(s-1)^3} \end{aligned}$$

1.4 Multiplication by t^n

If $L(f(t)) = \phi(s)$, then $L(t^n f(t)) = (-1)^n \frac{d^n}{ds^n}[\phi(s)]$ where n = 1, 2, 3,....

1. Find the Laplace transforms of following

(a) $te^{-t}\cos t$

(b) $t^2 sinat$

(c) $t\sin^2 3t$

Ans.

(a) We have

$$\begin{aligned} L(\cos t) &= \frac{s}{s^2+1} \\ L(t\cos t) &= -\frac{d}{ds}\left\{\frac{s}{s^2+1}\right\} \\ &= -\frac{(s^2+1) - s\cdot 2s}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2} \\ L(e^{-t}t\cos t) &= \frac{(s+1)^2-1}{((s+1)^2+1)^2} \end{aligned}$$

(b) We have

$$\begin{aligned} L(\sin at) &= \frac{a}{s^2+a^2} \\ L(t^2\sin at) &= (-1)^2\frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) \\ &= \frac{d}{ds}\left(\frac{-2as}{(s^2+a^2)^2}\right) \\ &= -2a\left(\frac{(s^2+a^2)^2 - s\times 2(s^2+a^2)\times 2s}{(s^2+a^2)^4}\right) \\ &= \frac{6as^2-2a^3}{(s^2+a^2)^3} \end{aligned}$$

(c) We have

$$\begin{aligned} L(\sin^2 3t) &= L(\frac{1-\cos 6t}{2} = \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+36}\right) \\ &= \frac{1}{2s} - \frac{s}{2(s^2+36)} \\ L(t\sin^2 3t) &= \frac{-d}{ds}\left(\frac{1}{2s} - \frac{s}{2(s^2+36)}\right) \\ &= \frac{1}{2s^2} + \frac{1}{2}\frac{s^2+36-s(2s)}{(s^2+36)^2} = \frac{1}{2s^2} + \frac{1}{2}\frac{36-s^2}{(s^2+36)^2} \end{aligned}$$

1.5 Division by t

If $L(f(t)) = \phi(s)$, then $L(\frac{f(t)}{t}) = \int_s^\infty \phi\, ds$

1. Find the Laplace transforms of following

   (a) $\frac{1-e^t}{t}$

   (b) $\frac{\sin t}{t}$

   (c) $\frac{e^{-t}\sin t}{t}$

   Ans.

   (a)

$$\begin{aligned} L(\frac{1-e^t}{t}) &= \int_s^\infty L(1-e^t)\, ds \\ &= \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds \\ &= [\log s - \log(s-1)]_s^\infty \\ &= \left[\log \frac{s}{s-1}\right]_s^\infty = \left[\log \frac{1}{1-\frac{1}{s}}\right]_s^\infty \\ &= \log 1 - \log \frac{s}{s-1} = \log \frac{s-1}{s} \end{aligned}$$

   (b)

$$\begin{aligned} L(\frac{\sin t}{t}) &= \int_s^\infty L(\sin t)\, ds \\ &= \int_s^\infty \frac{1}{s^2+1}\, ds = \left[\tan^{-1} s\right]_s^\infty \\ &= \tan^{-1}(\infty) - \tan^{-1}(s) = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s \end{aligned}$$

   (c)

$$\begin{aligned} L(\frac{\sin t}{t}) &= \int_s^\infty L(\sin t\, ds \\ &= \int_s^\infty \frac{1}{(s)^2+1}\, ds \\ &= \cot^{-1}(s) \text{ from above problem} \\ \therefore L(\frac{e^{-t}\sin t}{t}) &= \cot^{-1}(s+1) \end{aligned}$$

1.6 Transforms of Derivatives

If $f'(t)$ is continuous and $L(f(t)) = \phi(s)$, then $L(f'(t)) = s\phi(s) - f(0)$ provided $\lim_{x\to\infty} e^{-st} f(t) = 0$

Note:

$$\begin{aligned} L(f'') &= s^2L(f) - sf(0) - f'(0) \\ L(f''') &= s^3L(f) - s^2f(0) - sf'(0) - f''(0) \\ Ingeneral, L(f^{(n)}) &= s^nL(f) - s^{n-1}f(0) - s^{n-2}f'(0) - ..... - f^{(n-1)}(0) \end{aligned}$$

1.7 Transforms of Integrals

If $L(f(t)) = \phi(s)$, then $L(\int_0^t f(u)\,du) = \frac{\phi(s)}{s}$

1.8 Some Special Functions

Unit Step Function

The unit step function $u(t-a)$ is defined as $u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$, $a \geq 0$. The unit step function is also called the Heaviside function.

$$L(u(t-a)) = \frac{e^{-as}}{s}$$

Note:

Any piece wise continuous function $f(t) = \begin{cases} f_0(t) & 0 < t < t_1 \\ f_1(t) & t_1 < t < t_2 \\ f_2(t) & t_2 < t < t_3 \\ . & \\ . & \\ . & \\ f_{(n-1)}(t) & t_{(n-1)} < t < t_n \\ f_n(t) & t_n < t < \infty. \end{cases}$ defined on $0 < t < \infty$ can be given by the single expression

$f(t) = f_0(t)[u(t-0)-u(t-t_1)]+f_1(t)[u(t-t_1)-u(t-t_2)]+...+f_{(n-1)}(t)[u(t-t_{n-1})]+f_n(t)u(t-t_n)$.

1. Express the following function in terms of unit step function $f(t) = \begin{cases} 2+t^2 & if\, 0 < t < 2 \\ 6 & if\, 2 < t < 3 \\ \frac{2}{2t-5} & if\, t > 3 \end{cases}$

   Ans.

$$\begin{aligned} f(t) &= (2+t^2)[u(t-0) - u(t-2)] + 6[u(t-2) - u(t-3)] + \frac{2}{2t-5} \cdot u(t-3) \\ &= (2+t^2)u(t) + (4-t^2)u(t-2) + \left(\frac{32-12t}{2t-5}\right) \cdot u(t-3) \end{aligned}$$

Dirac delta function(Unit impulse function)

The unit impulse function denoted by $\delta(t)$ is defined by $\delta(t-a) = \begin{cases} 0, & t \neq 0 \\ \text{not defined} & t = 0. \end{cases}$

$$L(\delta(t-a)) = e^{-as}$$

1.9 Second shifting theorem

If f(t) has the Laplace transform $\phi(s)$ then $L(f(t-a)u(t-a)) = e^{-at}\phi(s)$

1. Find $L(\sin(t)u(t-\pi))$

$$\begin{aligned}\sin(t)u(t-\pi) &= \sin(t-\pi+\pi)u(t-\pi) = -\sin(t-\pi)u(t-\pi)\\ L(\sin(t)u(t-\pi)) &= -L(\sin(t-\pi)u(t-\pi))\\ &= -e^{-\pi s}L(\sin t) = -\frac{e^{-\pi s}}{s^2+1}\end{aligned}$$

2. Find the Laplace Transform of $(t-1)^2u(t-1)$

   Ans.

$$\begin{aligned}L((t-1)^2u(t-1)) &= e^{-s}L(t^2)\\ &= e^{-s}\frac{2}{s^3}\end{aligned}$$

3. Express th following function in terms of unit step function and hence find its Laplace transform $f(t) = \begin{cases} t^2 & 1 < t \leq 2 \\ 4t & t > 2 \end{cases}$

   Ans.

$$\begin{aligned}f(t) &= t^2[u(t-1)-u(t-2)] + 4t[u(t-2)]\\ &= t^2u(t-1) - t^2u(t-2) + 4tu(t-2)\\ &= (t-1+1)^2u(t-1) - (t-2+2)^2u(t-2) + 4(t-2+2)u(t-2)\\ &= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1) - (t-2)^2u(t-2)\\ &\quad -4(t-2)u(t-2) - 4u(t-2) + 4(t-2)u(t-2) + 8u(t-2)\\ &= (t-1)^2u(t-1) + 2(t-1)u(t-1) + u(t-1)\\ &\quad -(t-2)^2u(t-2) + 4u(t-2)\\ L(f(t)) &= e^{-s}L(t^2) + 2e^{-s}L(t) + e^{-s}L(1) - e^{-2s}L(t^2) + 4e^{-2s}L(t)\\ &= e^{-s}\frac{2}{s^3} + 2e^{-s}\frac{1}{s^2} + e^{-s}\frac{1}{s} - e^{-2s}\frac{2}{s^3} + 4e^{-2s}\frac{1}{s^2}\\ &= \frac{e^{-s}}{s^3}\left(s^2+2s+2\right) + \frac{e^{-2s}}{s^3}(4s-2)\end{aligned}$$

1.10 Exercise

1. Find $L(e^{-t}t^2)$
2. Find $L(e^{2t}\cos 3t)$
3. Find $L(\sinh at \cos bt)$
4. Find $L(e^{2t}\sin^2 3t)$
5. Find $L(t\cos 2t)$
6. Find $L(e^{-t}t\cos 2t)$
7. Find $L(\frac{1-\cos t}{t})$
8. Find $L(tu(t-2))$
9. Find $L(e^{-2t}u(t-1)$
10. Express th following function in terms of unit step function and hence find its Laplace transform
    $f(t) = \begin{cases} e^t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$

2 Inverse Laplace Transforms

If $L(f(t)) = \phi(s)$, then $L^{-1}[\phi(s)] = f(t)$, where L^{-1} is called the inverse Laplace transform operator.

Some basic Inverse Laplace Formulas

1. $L^{-1}[\frac{1}{s}] = 1$
2. $L^{-1}[\frac{1}{s^n}] = \frac{t^{n-1}}{(n-1)!}$
3. $L^{-1}[\frac{1}{s-a}] = e^{at}$
4. $L^{-1}[\frac{1}{s+a}] = e^{-at}$
5. $L^{-1}[\frac{1}{s^2+a^2}] = \frac{\sin at}{a}$
6. $L^{-1}[\frac{s}{s^2+a^2}] = \cos at$
7. $L^{-1}[\frac{1}{s^2-a^2}] = \frac{\sinh at}{a}$
8. $L^{-1}[\frac{s}{s^2-a^2}] = \cosh at$
9. $L^{-1}[\phi(s-a)] = e^{at}L^{-1}[\phi(s)]$ (Shifting property)
10. $L^{-1}[-\frac{d}{ds}L(f(t))] = tf(t)$
11. $L^{-1}[\int_0^{\infty} L(f(t))\, ds] = \frac{f(t)}{t}$

2.1 Inverse Transformation Using Partial Fraction

Some times a rational function of 's' can be expressed as sum of simple rational functions using partial fractions and then inverse transformed using shifting property.

S.No.	Form of the rational function	Form of the partial fraction
1.	$\frac{px+q}{(x-a)(x-b)}, a \neq b$	$\frac{A}{(x-a)} + \frac{B}{(x-b)}$
2.	$\frac{px+q}{(x-a)^2}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3.	$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{(x-a)} + \frac{B}{(x-b)} + \frac{C}{(x-c)}$
4.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
5.	$\frac{px^2+qx+r}{(x-a)^2(x-b)}$	$\frac{A}{(x-a)} + \frac{B}{(x-a)^2} + \frac{C}{(x-b)}$
6.	$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$ where x^2+bx+c cannot be factorised further	$\frac{A}{(x-a)} + \frac{Bx+C}{(x^2+bx+c)}$

1. Find the inverse Laplace transform of the following

   (a) $\frac{1}{s^5}$

   (b) $\frac{3s+2}{s^2+9}$

   (c) $\frac{5}{s^2+3s+7}$

   (d) $\frac{3s+2}{(s-1)(s^2+1)}$

   (e) $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$

Ans.

(a)

$$L^{-1}[\frac{1}{s^5}] = \frac{t^{5-1}}{(5-1)!} = \frac{t^4}{24}$$

(b)

$$\begin{aligned} L^{-1}\left[\frac{3s+2}{s^2+9}\right] &= 3L^{-1}\left[\frac{s}{s^2+9}\right] + \frac{2}{3}L^{-1}\left[\frac{3}{s^2+9}\right] \\ &= 3\cos 3t + \frac{2}{3}\sin 3t \end{aligned}$$

(c)

$$\begin{aligned} L^{-1}\left[\frac{5}{s^2+3s+7}\right] &= L^{-1}\left[\frac{5}{s^2+3s+7+(\frac{3}{2})^2-(\frac{3}{2})^2}\right] \\ &= L^{-1}\left[\frac{5}{(s+\frac{3}{2})^2+\frac{19}{2}}\right] \\ &= \frac{5}{\sqrt{\frac{19}{2}}}L^{-1}\left[\frac{\sqrt{\frac{19}{2}}}{(s+\frac{3}{2})^2+(\sqrt{\frac{19}{2}})^2}\right] \\ &= \frac{5\sqrt{2}}{\sqrt{19}}e^{-\frac{3t}{2}}L^{-1}\left(\frac{\sqrt{\frac{19}{2}}}{(s^2+\sqrt{\frac{19}{2}})^2}\right) \\ &= \frac{5\sqrt{2}}{\sqrt{19}}e^{-\frac{3t}{2}}\sin(\sqrt{\frac{19}{2}}t) \end{aligned}$$

(d)

We have $\dfrac{3s+2}{(s-1)(s^2+1)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+1}$

$\therefore 3s+2 = (s^2+1)A + (Bs+C)(s-1)$

Put s = 1 then, $5 = A(2) \implies A = \dfrac{5}{2}$

equating the term containing s^2 then $0 = A+B \implies B = -A = -\dfrac{5}{2}$

Put s = 0 then $2 = A + C \implies C = 2 - A = \dfrac{-1}{2}$

$$\begin{aligned} L^{-1}\left(\frac{3s+2}{(s-1)(s^2+1)}\right) &= L^{-1}\left(\frac{\frac{5}{2}}{s-1} + \frac{\frac{-5}{2}s+\frac{1}{2}}{s^2+1}\right) \\ &= \frac{5}{2}L^{-1}\left(\frac{1}{s-1}\right) - \frac{5}{2}L^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{2}L^{-1}\left(\frac{1}{s^2+1}\right) \\ &= \frac{5}{2}e^t - \frac{5}{2}\cos t + \frac{1}{2}\sin t \end{aligned}$$

(e)

$$\text{we have } \frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)} = \frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s-3}$$
$$ie,\ 2s^2-6s+5 = A(s-2)(s-3)+B(s-1)(s-3)+C(s-1)(s-2)$$

$$\begin{aligned} \text{Put s} = 1,\ 1 = A(2) &\implies A = \frac{1}{2} \\ \text{Put s} = 2,\ 1 = B(-1) &\implies B = -1 \\ \text{Put s} = 3,\ 5 = C(2) &\implies C = \frac{5}{2} \end{aligned}$$

$$\begin{aligned} L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] &= \frac{1}{2}L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s-2}\right) + \frac{5}{2}\left(\frac{1}{s-3}\right) \\ &= \frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t} \end{aligned}$$

2.2 Convolution

Given any two functions $f(t)$ and $g(t)$ defined for $t > 0$, their convolution is defined as the function $h(t)$, where $h(t) = \int_0^t f(u)g(t-u)\,du$ and is denoted by $f(t) * g(t)$

Laplace transform of convolution

If $h(t) = f(t) * g(t)$ then $L(h(t)) = L(f * g) = L(f(t))L(g(t))$

Note:

If $L^{-1}(\phi(s)) = f(t)$ and $L^{-1}(\psi(s)) = g(t)$ then $L^{-1}(\phi(s)\psi(s)) = f(t) * g(t)$

1. Use convolution theorem to find the inverse Laplace transform of $\frac{1}{(s-a)(s-b)}$.

$$\begin{aligned} L^{-1}\left(\frac{1}{(s-a)(s-b)}\right) &= L^{-1}\left(\frac{1}{s-a}\right) * L^{-1}\left(\frac{1}{s-b}\right) \\ &= e^{at} * e^{bt} = \int_0^t e^{au}e^{b(t-u)}\,du \\ &= \int_0^t e^{(a-b)t}e^{bt}\,du = e^{bt}\left[\frac{e^{(a-b)u}}{a-b}\right]_0^t \\ &= \frac{e^{bt}}{a-b}\left[e^{(a-b)t}-1\right] \\ &= \frac{e^{at}-e^{bt}}{a-b} \end{aligned}$$

2. Use convolution theorem to find the inverse Laplace transform of $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$.

$$\begin{aligned}
L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] &= L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+b^2}\right) \\
&= \cos at * \cos bt = \int_0^t \cos(au)\cos(b(t-u))\,du \\
&= \int_0^t \frac{1}{2}\left(\cos(bt+(a-b)u) + \cos((a+b)u-bt)\right)\,du \\
&= \frac{1}{2}\left[\frac{\sin(bt+(a-b)u)}{a-b} + \frac{\sin((a+b)u-bt)}{a+b}\right]_0^t \\
&= \frac{1}{2}\left(\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right) \\
&= \frac{1}{2}\left(\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right) \\
&= \frac{1}{2}\left(\frac{2a\sin at - 2b\sin bt}{(a-b)(a+b)}\right) \\
&= \frac{a\sin at - b\sin bt}{a^2-b^2}
\end{aligned}$$

3. Use convolution theorem to find the inverse Laplace transform of $\frac{1}{(s+1)(s-1)^2}$.

$$\begin{aligned}
L^{-1}\left[\frac{1}{(s+1)(s-1)^2}\right] &= L^{-1}\left(\frac{1}{s+1}\right) * \left(\frac{1}{(s-1)^2}\right) \\
&= e^{-t} * te^{t} = te^{t} * e^{-t} \\
&= \int_0^t ue^{u}e^{-(t-u)}\,du \\
&= e^{-t}\int_0^t ue^{2u}\,du \\
&= e^{-t}\left[(u)(\frac{e^{2u}}{2} - \frac{e^{2u}}{4})\right]_0^t \\
&= e^{-t}\left(\frac{te^{2t}}{2} - \frac{e^{2t}}{4} + \frac{1}{4}\right) \\
&= \frac{te^{t}}{2} - \frac{e^{t}}{4} + \frac{e^{-t}}{4}
\end{aligned}$$

2.3 Applications

Solution of Ordinary Differential Equations

Laplace transform converts an ordinary differential equation in the dependent variable y with a set of initial conditions into an algebraic equation in $L(y)$. Being an algebraic equation, the latter can be easily solved to get $L(y)$. Taking the inverse transform of $L(y)$, then yields the solution.

1. Using Laplace transform solve $y'' - 3y' + 2y = 4, \quad y(0) = 2, \quad y'(0) = 3$

Ans.

Given differential equation is $y'' - 3y' + 2y = 4$

Applying Laplace transform $L[y''] - 3L[y'] + 2L[y] = 4L[1]$

$$s^2L(y) - sy(0) - y'(0) - 3sL(y) - 3y(0) + 2L(y) = 4\frac{1}{s}$$

$$(s^2 - 3s + 2)L(y) - 2s - 3 + 6 = \frac{4}{s}$$

$$(s^2 - 3s + 2)L(y) = \frac{4}{s} + 2s - 3$$

$$L(y) = \frac{2s^2 - 3s + 4}{s(s^2 - 3s + 2)}$$

$$L(y) = \frac{2s^2 - 3s + 4}{s(s-1)(s-2)}$$

$$y = L^{-1}\left[\frac{2s^2 - 3s + 4}{s(s-1)(s-2)}\right] - - - -(1)$$

We have by partial fraction $\frac{2s^2 - 3s + 4}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2}$

$2s^2 - 3s + 4 = A(s-1)(s-2) + Bs(s-2) + Cs(s-1)$

$$\begin{aligned} \text{Put s = 0 } 4 &= A(2) \implies A = 2 \\ \text{Put s = 1 } 3 &= B(-1) \implies B = -3 \\ \text{put s = 2 } 6 &= C(2) \implies C = 3 \end{aligned}$$

$$\begin{aligned} \text{(1) becomes } y &= L^{-1}\left[\frac{2}{s} - \frac{3}{s-1} + \frac{3}{s-2}\right] \\ &= 2 - 3e^t + 3e^{2t} \end{aligned}$$

2. Using Laplace transform solve $y'' + 5y' + 6y = e^{-2t}$ Given $y(0) = y'(0) = 1$

Ans.

Given diff. equation is $y'' + 5y' + 6y = e^{-2t}$

Applying Laplace transform $L[y''] + 5L[y'] + 6L[y] = L[e^{-2t}$

$$s^2L[y] - sy(0) - y'(0) + 5sL[y] - 5y(0) + 6L[y] = \frac{1}{s+2}$$

$$(s^2 + 5s + 6)L(y) - s - 1 - 5 = \frac{1}{s+2}$$

$$(s^2 + 5s + 6)L(y) = \frac{1}{s+2} + s + 6$$

$$L(y) = \frac{s^2 + 8s + 13}{(s+2)(s^2 + 5s + 6)} = \frac{s^2 + 8s + 13}{(s+2)^2(s+3)} - - - -(1)$$

We have by partial fraction $\frac{s^2 + 8s + 13}{(s+2)^2(s+3)} = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{s+3}$

$s^2 + 8s + 13 = A(s+2)(s+3) + B(s+3) + C(s+2)^2$

$$\begin{aligned}
\text{Put s = -2 } 1 &= B(1) \implies B = 1\\
\text{Put s = -3 } -2 &= C(1) \implies C = -2\\
\text{Put s = 0 } 13 &= 6A + 3B + 4C \implies 6A = 13 - 3 + 8 \implies A = 3
\end{aligned}$$

$$\begin{aligned}
\text{(1) becomes } y &= L^{-1}\left[\frac{3}{s+2} + \frac{1}{(s+2)^2} - \frac{2}{s+3}\right]\\
&= 3e^{-2t} + te^{-2t} - 2e^{-3t}
\end{aligned}$$

3. Use Laplace transform solve $y'' + 2y' + 5y = e^{-t}\cos t$, given that $y(0) = 0$, $y'(0) = 1$

Given diff. equation is $y'' + 2y' + 5y = e^{-t}\cos t$

Applying Laplace transform $L(y'') + 2L(y') + 5L(y) = \dfrac{s+1}{(s+1)^2+1}$

$$s^2L(y) - sy(0) - y'(0) + 2sL(y) + 2y(0) + 5L(y) = \frac{s+1}{(s+1)^2+1}$$

$$(s^2 + 2s + 5)L(y) - 1 = \frac{s+1}{(s+1)^2+1}$$

$$(s^2 + 2s + 5)L(y) = \frac{s+1}{(s+1)^2+1} + 1$$

$$L(y) = \frac{s+1}{((s+1)^2+1)(s^2+2s+5)} + \frac{1}{s^2+2s+5}$$

$$L(y) = \frac{s+1}{((s+1)^2+1)(s^2+2s+1+4)} + \frac{1}{(s^2+2s+1+4)}$$

$$y = L^{-1}\left[\frac{s+1}{((s+1)^2+1)((s+1)^2+2^2)} + \frac{1}{((s+1)^2+2^2)}\right]$$

$$y = e^{-t}L^{-1}\left[\frac{s}{(s^2+1)(s^2+2^2)} + \frac{1}{s^2+2^2}\right]$$

But we have $(s^2+2^2) - (s^2+1) = 3$

$$y = e^{-t}\left[L^{-1}\left(\frac{s[(s^2+2^2) - (s^2+1)]}{s(s^2+1)(s^2+2^2)}\right) + L^{-1}\left(\frac{1}{s^2+2^2}\right)\right]$$

$$y = e^{-t}\left[\frac{1}{3}L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{s}{s^2+2^2}\right)\right] + e^{-t}\frac{\sin 2t}{2}$$

$$= \frac{e^{-t}}{3}\cos t - \frac{e^{-t}}{3}\cos 2t + \frac{e^{-t}}{2}\sin 2t$$

$$= e^{-t}\left(\frac{\cos t}{3} - \frac{\cos 2t}{3} + \frac{\sin 2t}{2}\right)$$

2.4 Exercise

1. Find the inverse Laplace transforms of the following

(a) $\frac{3s-2}{s^2-5s+6}$

(b) $\frac{2s-3}{s^2+6s+13}$

(c) $\frac{6s+7}{(s+2)(s-1)^2}$

(d) $\frac{2s+5}{(s+2)(s^2+9)}$

(e) $\frac{s}{s^3+1}$

2. Use Convolution theorem find the inverse of the following

   (a) $\frac{s}{(s^2+1)^2}$

   (b) $\frac{1}{s(s+1)}$

   (c) $\frac{1}{(s^2+1)^2}$

   (d) $\frac{1}{s(s-3)^2}$

   (e) $\frac{s}{(s+2)(s^2+9)}$

3. Using Laplace transform solve following differential equations

   (a) $y'' - 2y' - 3y = \sin t$, given $y(0) = y'(0) = 0$

   (b) $y'' - 3y' + 2y = 4t + e^{3t}$, given that $y(0) = 1, \quad y'(0) = -1$

   (c) $y''' - 3y'' + 3y' - y = t^2e^t$, given $y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$

   (d) $y'' - 3y' + 2y = 4$, given that $y(0) = 2, \quad y'(0) = 3$

   (e) $y'' + 2y' + 6y = 6te^{-t}$, given that $y(0) = 2 \quad y'(0) = 5$